Linear Programming

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Outline of the Lecture

1. Linear Optimization Problems
2. A Geometric View on Linear Programming
3. Standard Form for a Linear Minimization
4. Application: The Diet Problem
5. The Simplex Algorithm - the idea
6. Using `linprog` in MATLAB
7. Duality Theory

1. Optimization Problems

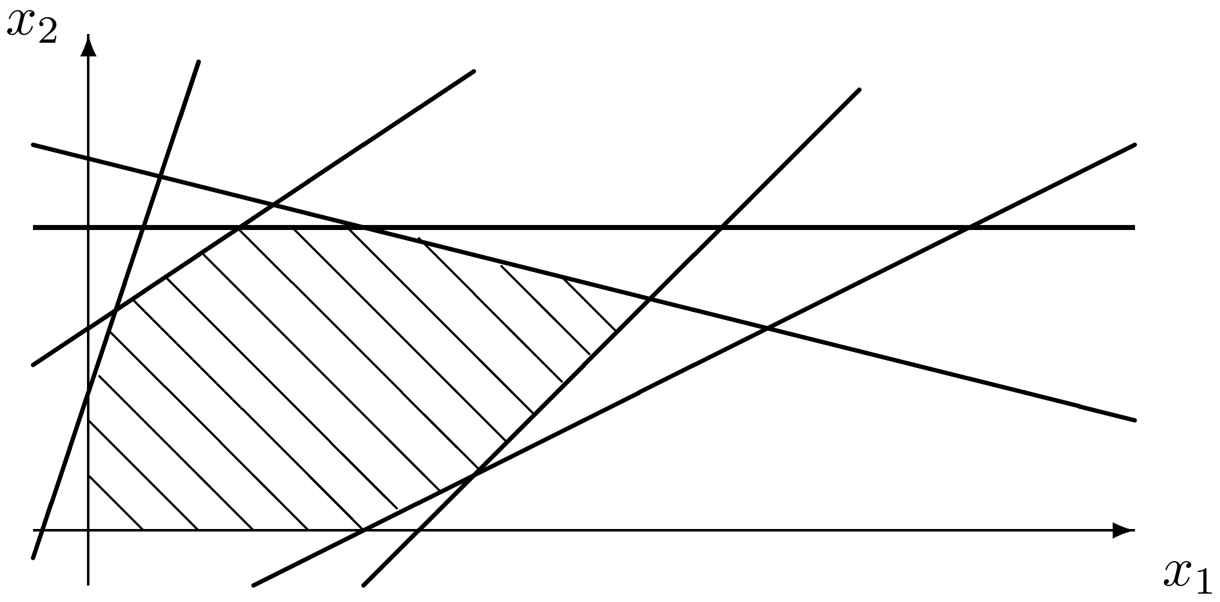
maximize profit		minimize cost
subject to		subject to
$E(\mathbf{x}) = \mathbf{a}$	*equalities*	$E(\mathbf{x}) = \mathbf{a}$
$I(\mathbf{x}) \leq \mathbf{b}$	*inequalities*	$I(\mathbf{x}) \geq \mathbf{b}$

A Linear Programming problem is an optimization problem

- with one linear objective function; and
- all the constraints (equalities and inequalities) are linear.

2. A Geometric View on Linear Programming

The linear constraints define a polyhedron:



- polyhedron is nonempty: problem is *feasible*
- polyhedron is bounded: problem is *well-posed*

3. Standard Form for a Linear Minimization

consider $\min \mathbf{c}^T\mathbf{x}$ $\qquad \mathbf{c}, \mathbf{x} \in \mathbb{R}^n$

subject to $A\mathbf{x} \geq \mathbf{b}$ $\qquad A \in \mathbb{R}^{m\times n}, \mathbf{b} \in \mathbb{R}^m$

slack variables $\mathbf{z} \in \mathbb{R}^m$: $A\mathbf{x} \geq \mathbf{b}$ becomes $A\mathbf{x} - \mathbf{z} = \mathbf{b},\ \mathbf{z} \geq \mathbf{0}$

splitting $\mathbf{x} = \mathbf{x}^+ - \mathbf{x}^-$, $\mathbf{x}^+ \geq \mathbf{0}$, $\mathbf{x}^- \geq \mathbf{0}$:

$$\min\ [\mathbf{c}\ \ -\mathbf{c}\ \ \mathbf{0}] \begin{bmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \\ \mathbf{z} \end{bmatrix}$$

$$\text{subject to } [A\ \ -A\ \ -I] \begin{bmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \\ \mathbf{z} \end{bmatrix} = \mathbf{b}, \qquad \begin{bmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \\ \mathbf{z} \end{bmatrix} \geq \mathbf{0}$$

4. Application: the Diet Problem

Problem: plan healthy but economical meals.

nutritional elements	food quantities x_1	x_2	$\cdots$	x_n	minimum requirements
1	a_{11}	a_{12}	$\cdots$	a_{1n}	c_1
2	a_{21}	a_{22}	$\cdots$	a_{2n}	c_2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
m	a_{m1}	a_{m2}	$\cdots$	a_{mn}	c_m
price	p_1	p_2	$\cdots$	p_n	

Solve a linear programming problem (typically $m > n$) :

$$\min\ p_1x_1 + p_2x_2 + \cdots + p_nx_n$$

$$\text{subject to } a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq c_i,\ i = 1, 2, \ldots, m$$

5. The Simplex Algorithm

Idea: move from one vertex to another vertex,
improving the objective value in each move.

Step 0: Bring the problem in standard form:
min $\mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$.

Step 1: Find one feasible point.

Step 2: Find one feasible *vertex* point.

Step 3: Move to vertex with improved objective value,
repeat until at optimum.

6. Using `linprog` in MATLAB

- MATLAB has an optimization toolbox

  type `help toolbox\optim` to see an overview

- type `help linprog` for information

  about the use of the simplex algorithm in MATLAB :

  `x = linprog(f,A,b)` solves

  ```
        min f'*x    subject to   A*x <= b
         x
  ```

- for the diet problem: `A*x >= c` ⇔ `-A*x <= -c`

7. Every LP problem has a dual problem

$$\max \sum_{j=1}^{n} c_j x_j \qquad\qquad \min \sum_{i=1}^{m} b_i y_i$$

subject to

$$\begin{cases} \displaystyle\sum_{j=1}^{n} a_{ij} x_j \le b_i, & {}_{i=1,2,\ldots,m} \\ x_j \ge 0, & {}_{j=1,2,\ldots,n} \end{cases}$$

subject to

$$\begin{cases} \displaystyle\sum_{i=1}^{m} a_{ij} y_j \ge c_j, & {}_{j=1,2,\ldots,n} \\ y_i \ge 0, & {}_{i=1,2,\ldots,m} \end{cases}$$

economic interpretation :

x_j : level of activity j

c_j : unit profit of activity j

b_i : amount available of resource i

a_{ij} : amount of resource i consumed by each unit of activity j

y_i : contribution to profit per unit of resource i