Alpha Theory to Certify Roots

MCS 563 Lecture 5
Analytic Symbolic Computation
Jan Verschelde, 21 January 2011

Alpha Theory to Certify Roots

approximate zeroes

For a function f in one variable x, Newton's method defines

$$x_{k+1} = N_f(x_k), \quad k = 0, 1, \ldots \text{ and } N_f(x) = x - \frac{f(x)}{f'(x)},$$

provided the derivative $f'(x_k) \neq 0$ for all k.

z is ***an approximate zero for a root ζ of f*** ($f(\zeta) = 0$) if

1. $x_{k+1} = N_f(x_k)$ is defined for all $k = 0, 1, \ldots$ starting with $x_0 = z$; and
2. the convergence is quadratic:

$$|x_k - \zeta| \leq \left(\frac{1}{2}\right)^{2^k - 1} |z - \zeta|.$$

Needed: a criterion to determine if z is an approximate zero.

approximate zeroes

For a function f in one variable x, Newton's method defines

$$x_{k+1} = N_f(x_k), \quad k = 0, 1, \ldots \text{ and } N_f(x) = x - \frac{f(x)}{f'(x)},$$

provided the derivative $f'(x_k) \neq 0$ for all k.

z is ***an approximate zero for a root*** ζ ***of*** f ($f(\zeta) = 0$) if

1. $x_{k+1} = N_f(x_k)$ is defined for all $k = 0, 1, \ldots$ starting with $x_0 = z$; and
2. the convergence is quadratic:

$$|x_k - \zeta| \leq \left(\frac{1}{2}\right)^{2^k - 1} |z - \zeta|.$$

Needed: a criterion to determine if z is an approximate zero.

approximate zeroes

For a function f in one variable x, Newton's method defines

$$x_{k+1} = N_f(x_k), \quad k = 0, 1, \ldots \text{ and } N_f(x) = x - \frac{f(x)}{f'(x)},$$

provided the derivative $f'(x_k) \neq 0$ for all k.

z is ***an approximate zero for a root ζ of f*** ($f(\zeta) = 0$) if

1. $x_{k+1} = N_f(x_k)$ is defined for all $k = 0, 1, \ldots$ starting with $x_0 = z$; and
2. the convergence is quadratic:

$$|x_k - \zeta| \leq \left(\frac{1}{2}\right)^{2^k-1} |z - \zeta|.$$

Needed: a criterion to determine if z is an approximate zero.

the gamma function

For a criterion to determine if a given number is an approximate zero, we need an auxiliary function

$$\begin{array}{rcccl} \gamma & : & \mathbb{C}[x] \times \mathbb{C} & \rightarrow & \mathbb{R} \\ & & (f, x) & \mapsto & \gamma(f,x) = \displaystyle \max_{k=2}^{\deg(f)} \left| \frac{f^{(k)}(x)}{k!\, f'(x)} \right|^{1/(k-1)} \end{array}$$

where $f^{(k)}$ is the kth derivative of f.

Note: $\gamma(f,x)$ is undefined for $f'(x) = 0$.

Example: $\gamma(x^3 - 1, x) = \max \left(\left| \frac{6x}{2!\, 3x^2} \right|, \left| \frac{6}{3!\, 3x^2} \right|^{1/2} \right)$

is defined for all $x \neq 0$.

the gamma function

For a criterion to determine if a given number is an approximate zero, we need an auxiliary function

$$\begin{array}{rcccc} \gamma & : & \mathbb{C}[x] \times \mathbb{C} & \rightarrow & \mathbb{R} \\ & & (f, x) & \mapsto & \gamma(f, x) = \max\limits_{k=2}^{\deg(f)} \left| \dfrac{f^{(k)}(x)}{k!\, f'(x)} \right|^{1/(k-1)} \end{array}$$

where $f^{(k)}$ is the kth derivative of f.

Note: $\gamma(f, x)$ is undefined for $f'(x) = 0$.

Example: $\gamma(x^3 - 1, x) = \max \left(\left| \frac{6x}{2!\ 3x^2} \right|, \left| \frac{6}{3!\ 3x^2} \right|^{1/2} \right)$
is defined for all $x \neq 0$.

Alpha Theory to Certify Roots

a first criterion

If we know the exact root ζ, then a criterion is below.

Theorem 1

Assume $f(\zeta) = 0$ and $f'(\zeta) \neq 0$.

If $|x - \zeta| \leq \dfrac{3 - \sqrt{7}}{2\gamma(f, \zeta)}$, then x is an approximate zero for ζ.

The number $\dfrac{3 - \sqrt{7}}{2\gamma(f, \zeta)}$ defines the radius of a disk

- in the complex plane, centered at ζ
- for which Newton's method converges quadratically.

a first criterion

If we know the exact root ζ, then a criterion is below.

Theorem 1

Assume $f(\zeta) = 0$ and $f'(\zeta) \neq 0$.

If $|x - \zeta| \leq \dfrac{3 - \sqrt{7}}{2\gamma(f, \zeta)}$, then x is an approximate zero for ζ.

The number $\dfrac{3 - \sqrt{7}}{2\gamma(f, \zeta)}$ defines the radius of a disk

- in the complex plane, centered at ζ
- for which Newton's method converges quadratically.

more auxiliary functions

For a criterion computation at a point x itself, we need two more auxiliary functions:

$$\begin{array}{rcccl} \beta & : & \mathbb{C}[x] \times \mathbb{C} & \rightarrow & \mathbb{R} \\ & & (f, x) & \mapsto & \beta(f, x) = |x - N_f(x)| = \left| \frac{f(x)}{f'(x)} \right| \end{array}$$

and

$$\begin{array}{rcccl} \alpha & : & \mathbb{C}[x] \times \mathbb{C} & \rightarrow & \mathbb{R} \\ & & (f, x) & \mapsto & \alpha(f, x) = \beta(f, x)\gamma(f, x). \end{array}$$

Once more, we assume $f'(x) \neq 0$.

more auxiliary functions

For a criterion computation at a point x itself,
we need two more auxiliary functions:

$$\begin{array}{rcccl} \beta & : & \mathbb{C}[x] \times \mathbb{C} & \rightarrow & \mathbb{R} \\ & & (f, x) & \mapsto & \beta(f, x) = |x - N_f(x)| = \left| \frac{f(x)}{f'(x)} \right| \end{array}$$

and

$$\begin{array}{rcccl} \alpha & : & \mathbb{C}[x] \times \mathbb{C} & \rightarrow & \mathbb{R} \\ & & (f, x) & \mapsto & \alpha(f, x) = \beta(f, x)\gamma(f, x). \end{array}$$

Once more, we assume $f'(x) \neq 0$.

the criterion at a point

To determine whether x is an approximate zero for f:

Theorem 2

There is a universal constant α_0:
if $\alpha(f,x) < \alpha_0$ then

- *x is an approximate zero for a root ζ of f and*
- *$|x - \zeta| \leq 2\beta(f,x)$.*

In [Blum, Cucker, Shub, Smale 1998], the best value for $\alpha_0 = \frac{1}{4}(13 - 3\sqrt{17}) \approx 0.157671$.

Alpha Theory to Certify Roots

the telescope system

Consider

$$f(\mathbf{x}) = \begin{cases} x_1 - \alpha = 0 \\ x_2 - x_1^2 = 0 \\ x_3 - x_2^2 = 0 \\ x_4 - x_3^2 = 0 \end{cases}$$

where α is a parameter, with $(\alpha, \alpha^2, \alpha^4, \alpha^8)$ as solution.

Although $f(\mathbf{x}) = \mathbf{0}$ consists only of quadrics,
for $\alpha < 1$ or $\alpha > 1$, the solution decreases or increases.

The telescoping poses challenges for numerical methods.

Alpha Theory to Certify Roots

the first lemma

Lemma 1

For a polynomial f in x, $\deg(f) = d$, $f(\zeta) = 0$, $f'(\zeta) \neq 0$, we have:

$$|N_f(x) - \zeta| \leq \left|\frac{f'(\zeta)}{f(x)}\right| \left[\sum_{k=2}^{d} (k-1)\left(\gamma(f,\zeta)|x-\zeta|\right)^{k-1}\right] |x - \zeta|.$$

Proof. Taylor series of f about ζ, using $f(\zeta) = 0$:

$$f(x) = \sum_{k=1}^{d} \frac{f^{(k)}(\zeta)}{k!}(x-\zeta)^k \text{ and } f'(x) = \sum_{k=1}^{d} \frac{f^{(k)}(\zeta)}{(k-1)!}(x-\zeta)^{k-1}.$$

As $\frac{1}{(k-1)!} - \frac{1}{k!} = \frac{k-1}{k!}$:

$$(x-\zeta)f'(x) - f(x) = \sum_{k=1}^{d} (k-1)\frac{f^{(k)}(\zeta)}{k!}(x-\zeta)^k.$$

the first lemma

Lemma 1

For a polynomial f in x, $\deg(f) = d$, $f(\zeta) = 0$, $f'(\zeta) \neq 0$, we have:

$$|N_f(x) - \zeta| \leq \left| \frac{f'(\zeta)}{f(x)} \right| \left[\sum_{k=2}^{d} (k-1) \left(\gamma(f,\zeta) |x - \zeta| \right)^{k-1} \right] |x - \zeta|.$$

Proof. Taylor series of f about ζ, using $f(\zeta) = 0$:

$$f(x) = \sum_{k=1}^{d} \frac{f^{(k)}(\zeta)}{k!} (x - \zeta)^k \ \text{ and } \ f'(x) = \sum_{k=1}^{d} \frac{f^{(k)}(\zeta)}{(k-1)!} (x - \zeta)^{k-1}.$$

As $\frac{1}{(k-1)!} - \frac{1}{k!} = \frac{k-1}{k!}$:

$$(x - \zeta) f'(x) - f(x) = \sum_{k=1}^{d} (k-1) \frac{f^{(k)}(\zeta)}{k!} (x - \zeta)^k.$$

the first lemma

Lemma 1

For a polynomial f in x, $\deg(f) = d$, $f(\zeta) = 0$, $f'(\zeta) \neq 0$, we have:

$$|N_f(x) - \zeta| \leq \left| \frac{f'(\zeta)}{f(x)} \right| \left[\sum_{k=2}^{d} (k-1) \left(\gamma(f,\zeta) |x - \zeta| \right)^{k-1} \right] |x - \zeta|.$$

Proof. Taylor series of f about ζ, using $f(\zeta) = 0$:

$$f(x) = \sum_{k=1}^{d} \frac{f^{(k)}(\zeta)}{k!} (x - \zeta)^k \text{ and } f'(x) = \sum_{k=1}^{d} \frac{f^{(k)}(\zeta)}{(k-1)!} (x - \zeta)^{k-1}.$$

As $\frac{1}{(k-1)!} - \frac{1}{k!} = \frac{k-1}{k!}$:

$$(x - \zeta) f'(x) - f(x) = \sum_{k=1}^{d} (k-1) \frac{f^{(k)}(\zeta)}{k!} (x - \zeta)^k.$$

proof continued

Recall $N_f(x) = x - \frac{f(x)}{f'(x)}$, so we have:

$$N_f(x) - \zeta = \frac{1}{f'(x)} \left[\sum_{k=2}^{d} (k-1) \frac{f^{(k)}(\zeta)}{k!} (x-\zeta)^{k-1} \right] (x-\zeta)$$

$$= \frac{f'(\zeta)}{f'(x)} \left[\sum_{k=2}^{d} (k-1) \frac{f^{(k)}(\zeta)}{k! f'(\zeta)} (x-\zeta)^{k-1} \right] (x-\zeta).$$

By repeated application of $|a+b| \leq |a| + |b|$:

$$|N_f(x) - \zeta| \leq \left| \frac{f'(\zeta)}{f'(x)} \right| \left[\sum_{k=2}^{d} (k-1) \left(\left| \frac{f^{(k)}(\zeta)}{k! f'(\zeta)} \right|^{1/(k-1)} |x-\zeta| \right)^{k-1} \right]$$

The lemma follows from the definition of $\gamma(f,x)$, for $x = \zeta$.

proof continued

Recall $N_f(x) = x - \frac{f(x)}{f'(x)}$, so we have:

$$
\begin{aligned}
N_f(x) - \zeta &= \frac{1}{f'(x)} \left[\sum_{k=2}^{d} (k-1) \frac{f^{(k)}(\zeta)}{k!} (x-\zeta)^{k-1} \right] (x-\zeta) \\
&= \frac{f'(\zeta)}{f'(x)} \left[\sum_{k=2}^{d} (k-1) \frac{f^{(k)}(\zeta)}{k!f'(\zeta)} (x-\zeta)^{k-1} \right] (x-\zeta).
\end{aligned}
$$

By repeated application of $|a+b| \leq |a| + |b|$:

$$
|N_f(x) - \zeta| \leq \left| \frac{f'(\zeta)}{f'(x)} \right| \left[\sum_{k=2}^{d} (k-1) \left(\left| \frac{f^{(k)}(\zeta)}{k!f'(\zeta)} \right|^{1/(k-1)} |x-\zeta| \right)^{k-1} \right]
$$

The lemma follows from the definition of $\gamma(f,x)$, for $x = \zeta$.

proof continued

Recall $N_f(x) = x - \frac{f(x)}{f'(x)}$, so we have:

$$\begin{aligned} N_f(x) - \zeta &= \frac{1}{f'(x)} \left[\sum_{k=2}^{d} (k-1) \frac{f^{(k)}(\zeta)}{k!} (x-\zeta)^{k-1} \right] (x-\zeta) \\ &= \frac{f'(\zeta)}{f'(x)} \left[\sum_{k=2}^{d} (k-1) \frac{f^{(k)}(\zeta)}{k! f'(\zeta)} (x-\zeta)^{k-1} \right] (x-\zeta). \end{aligned}$$

By repeated application of $|a+b| \leq |a| + |b|$:

$$|N_f(x) - \zeta| \leq \left| \frac{f'(\zeta)}{f'(x)} \right| \left[\sum_{k=2}^{d} (k-1) \left(\left| \frac{f^{(k)}(\zeta)}{k! f'(\zeta)} \right|^{1/(k-1)} |x-\zeta| \right)^{k-1} \right]$$

The lemma follows from the definition of $\gamma(f,x)$, for $x = \zeta$.

the second lemma

Lemma 2

For a polynomial f in x, $\deg(f) = d$, $f(\zeta) = 0$, $f'(\zeta) \neq 0$, we have:

$$|N_f(x) - \zeta| \leq \left| \frac{f'(\zeta)}{f(x)} \right| \frac{u}{(1-u)^2} |x - \zeta|, \quad \text{for } u = \gamma(f,\zeta)|x-\zeta| < 1.$$

Proof. $\displaystyle\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ and $\displaystyle\sum_{k=1}^{\infty} k r^{k-1} = \frac{1}{(1-r)^2}$, $0 \leq r < 1$.

$$\text{For } u < 1 : \sum_{k=2}^{d} (k-1) \left(\gamma(f,\zeta)|x-\zeta| \right)^{k-1} \leq \sum_{k=1}^{\infty} (k-1) u^{k-1}$$

$$= \sum_{k=1}^{\infty} k u^{k-1} - \sum_{k=1}^{\infty} u^{k-1} = \frac{1}{(1-u)^2} - \frac{1}{1-u} = \frac{u}{(1-u)^2}.$$

the second lemma

Lemma 2

For a polynomial f in x, $\deg(f) = d$, $f(\zeta) = 0$, $f'(\zeta) \neq 0$, we have:

$$|N_f(x) - \zeta| \leq \left| \frac{f'(\zeta)}{f(x)} \right| \frac{u}{(1-u)^2} |x - \zeta|, \quad \textit{for } u = \gamma(f,\zeta)|x-\zeta| < 1.$$

Proof. $\displaystyle\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ and $\displaystyle\sum_{k=1}^{\infty} k r^{k-1} = \frac{1}{(1-r)^2}$, $0 \leq r < 1$.

For $u < 1$: $\displaystyle\sum_{k=2}^{d} (k-1)\left(\gamma(f,\zeta)|x-\zeta|\right)^{k-1} \leq \sum_{k=1}^{\infty} (k-1) u^{k-1}$

$$= \sum_{k=1}^{\infty} k u^{k-1} - \sum_{k=1}^{\infty} u^{k-1} = \frac{1}{(1-u)^2} - \frac{1}{1-u} = \frac{u}{(1-u)^2}.$$

the second lemma

Lemma 2

For a polynomial f in x, $\deg(f) = d$, $f(\zeta) = 0$, $f'(\zeta) \neq 0$, we have:

$$|N_f(x) - \zeta| \leq \left| \frac{f'(\zeta)}{f(x)} \right| \frac{u}{(1-u)^2} |x - \zeta|, \quad \textit{for } u = \gamma(f,\zeta)|x-\zeta| < 1.$$

Proof. $\displaystyle\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ and $\displaystyle\sum_{k=1}^{\infty} k r^{k-1} = \frac{1}{(1-r)^2}$, $0 \leq r < 1$.

For $u < 1$: $\displaystyle\sum_{k=2}^{d} (k-1)\left(\gamma(f,\zeta)|x-\zeta|\right)^{k-1} \leq \sum_{k=1}^{\infty} (k-1) u^{k-1}$

$$= \sum_{k=1}^{\infty} k u^{k-1} - \sum_{k=1}^{\infty} u^{k-1} = \frac{1}{(1-u)^2} - \frac{1}{1-u} = \frac{u}{(1-u)^2}.$$

the second lemma

Lemma 2

For a polynomial f in x, $\deg(f) = d$, $f(\zeta) = 0$, $f'(\zeta) \neq 0$, we have:

$$|N_f(x) - \zeta| \leq \left| \frac{f'(\zeta)}{f(x)} \right| \frac{u}{(1-u)^2} |x - \zeta|, \quad \textit{for } u = \gamma(f, \zeta)|x - \zeta| < 1.$$

Proof. $\displaystyle\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ and $\displaystyle\sum_{k=1}^{\infty} k r^{k-1} = \frac{1}{(1-r)^2}$, $0 \leq r < 1$.

$$\text{For } u < 1: \sum_{k=2}^{d} (k-1) \left(\gamma(f,\zeta)|x - \zeta| \right)^{k-1} \leq \sum_{k=1}^{\infty} (k-1) u^{k-1}$$

$$= \sum_{k=1}^{\infty} k u^{k-1} - \sum_{k=1}^{\infty} u^{k-1} = \frac{1}{(1-u)^2} - \frac{1}{1-u} = \frac{u}{(1-u)^2}.$$

the third lemma

Lemma 3

For a polynomial f in x, $\deg(f) = d$, $f(\zeta) = 0$, $f'(\zeta) \neq 0$, *we have:*

$$|N_f(x) - \zeta| \leq \left(\frac{u}{1 - 4u + 2u^2} \right) |x - \zeta|,$$

for

$$u = \gamma(f, \zeta)|x - \zeta| < 1 - \frac{\sqrt{2}}{2}.$$

Proof. We need to show $\left| \frac{f'(\zeta)}{f(x)} \right| \leq \frac{(1-u)^2}{1-4u+2u^2}$.

the third lemma

Lemma 3

For a polynomial f in x, $\deg(f) = d$, $f(\zeta) = 0$, $f'(\zeta) \neq 0$, *we have:*

$$|N_f(x) - \zeta| \leq \left(\frac{u}{1 - 4u + 2u^2} \right) |x - \zeta|,$$

for

$$u = \gamma(f, \zeta)|x - \zeta| < 1 - \frac{\sqrt{2}}{2}.$$

Proof. We need to show $\left| \frac{f'(\zeta)}{f(x)} \right| \leq \frac{(1-u)^2}{1-4u+2u^2}$.

rewriting $f'(x)/f'(\zeta)$

$$
\begin{aligned}
\frac{f'(x)}{f'(\zeta)} &= \frac{1}{f'(\zeta)}\left[\sum_{k=1}^{d} \frac{f^{(k)}(\zeta)}{(k-1)!}(x-\zeta)^{k-1}\right] \\
&= \frac{1}{f'(\zeta)}\left[f'(\zeta) + \sum_{k=2}^{d} \frac{f^{(k)}(\zeta)}{(k-1)!}(x-\zeta)^{k-1}\right] \\
&= 1 + \sum_{k=2}^{d} \frac{f^{(k)}(\zeta)}{(k-1)!f'(\zeta)}(x-\zeta)^{k-1} \\
&= 1 + \underbrace{\sum_{k=2}^{d} k\left(\left(\frac{f^{(k)}(\zeta)}{k!f'(\zeta)}\right)^{1/(k-1)}(x-\zeta)\right)^{k-1}}_{=B} \\
&= 1 + B.
\end{aligned}
$$

rewriting $f'(x)/f'(\zeta)$

$$\begin{aligned}
\frac{f'(x)}{f'(\zeta)} &= \frac{1}{f'(\zeta)}\left[\sum_{k=1}^{d}\frac{f^{(k)}(\zeta)}{(k-1)!}(x-\zeta)^{k-1}\right] \\
&= \frac{1}{f'(\zeta)}\left[f'(\zeta)+\sum_{k=2}^{d}\frac{f^{(k)}(\zeta)}{(k-1)!}(x-\zeta)^{k-1}\right] \\
&= 1+\sum_{k=2}^{d}\frac{f^{(k)}(\zeta)}{(k-1)!f'(\zeta)}(x-\zeta)^{k-1} \\
&= 1+\underbrace{\sum_{k=2}^{d}k\left(\left(\frac{f^{(k)}(\zeta)}{k!f'(\zeta)}\right)^{1/(k-1)}(x-\zeta)\right)^{k-1}}_{=B} \\
&= 1+B.
\end{aligned}$$

rewriting $f'(x)/f'(\zeta)$

$$
\begin{aligned}
\frac{f'(x)}{f'(\zeta)} &= \frac{1}{f'(\zeta)}\left[\sum_{k=1}^{d} \frac{f^{(k)}(\zeta)}{(k-1)!}(x-\zeta)^{k-1}\right] \\
&= \frac{1}{f'(\zeta)}\left[f'(\zeta) + \sum_{k=2}^{d} \frac{f^{(k)}(\zeta)}{(k-1)!}(x-\zeta)^{k-1}\right] \\
&= 1 + \sum_{k=2}^{d} \frac{f^{(k)}(\zeta)}{(k-1)!f'(\zeta)}(x-\zeta)^{k-1} \\
&= 1 + \underbrace{\sum_{k=2}^{d} k\left(\left(\frac{f^{(k)}(\zeta)}{k!f'(\zeta)}\right)^{1/(k-1)}(x-\zeta)\right)^{k-1}}_{=B} \\
&= 1 + B.
\end{aligned}
$$

rewriting $f'(x)/f'(\zeta)$

$$
\begin{aligned}
\frac{f'(x)}{f'(\zeta)} &= \frac{1}{f'(\zeta)}\left[\sum_{k=1}^{d} \frac{f^{(k)}(\zeta)}{(k-1)!}(x-\zeta)^{k-1}\right] \\
&= \frac{1}{f'(\zeta)}\left[f'(\zeta) + \sum_{k=2}^{d} \frac{f^{(k)}(\zeta)}{(k-1)!}(x-\zeta)^{k-1}\right] \\
&= 1 + \sum_{k=2}^{d} \frac{f^{(k)}(\zeta)}{(k-1)!f'(\zeta)}(x-\zeta)^{k-1} \\
&= 1 + \underbrace{\sum_{k=2}^{d} k\left(\left(\frac{f^{(k)}(\zeta)}{k!f'(\zeta)}\right)^{1/(k-1)}(x-\zeta)\right)^{k-1}}_{=B} \\
&= 1 + B.
\end{aligned}
$$

estimating $f'(\zeta)/f'(x)$

$$\left| \frac{f'(\zeta)}{f'(x)} \right| = \frac{1}{|1+B|} \leq \sum_{k=0} |B|^k = \frac{1}{1-|B|},$$

provided $|B| < 1$. Therefore we estimate $|B|$:

$$|B| \leq \sum_{k=2}^{d} k u^{k-1} = \sum_{k=1}^{d} k u^{k-1} - 1 \leq \sum_{k=1}^{\infty} k u^{k-1} - 1 = \frac{1}{(1-u)^2} - 1.$$

To verify whether $|B| < 1$ we solve $\frac{1}{(1-u)^2} - 1 = 1$.
The smallest positive root of $1 - 4u + 2u^2$ is $1 - \frac{\sqrt{2}}{2}$,
so $u < 1 - \frac{\sqrt{2}}{2} \Rightarrow |B| < 1$. Thus:

$$\left| \frac{f'(\zeta)}{f'(x)} \right| \leq \frac{1}{1 - \left(\frac{1}{(1-u)^2} - 1 \right)} = \frac{(1-u)^2}{1 - 4u + 2u^2}$$

estimating $f'(\zeta)/f'(x)$

$$\left| \frac{f'(\zeta)}{f'(x)} \right| = \frac{1}{|1+B|} \leq \sum_{k=0} |B|^k = \frac{1}{1-|B|},$$

provided $|B| < 1$. Therefore we estimate $|B|$:

$$|B| \leq \sum_{k=2}^{d} k u^{k-1} = \sum_{k=1}^{d} k u^{k-1} - 1 \leq \sum_{k=1}^{\infty} k u^{k-1} - 1 = \frac{1}{(1-u)^2} - 1.$$

To verify whether $|B| < 1$ we solve $\frac{1}{(1-u)^2} - 1 = 1$.
The smallest positive root of $1 - 4u + 2u^2$ is $1 - \frac{\sqrt{2}}{2}$,
so $u < 1 - \frac{\sqrt{2}}{2} \Rightarrow |B| < 1$. Thus:

$$\left| \frac{f'(\zeta)}{f'(x)} \right| \leq \frac{1}{1 - \left(\frac{1}{(1-u)^2} - 1 \right)} = \frac{(1-u)^2}{1 - 4u + 2u^2}$$

estimating $f'(\zeta)/f'(x)$

$$\left| \frac{f'(\zeta)}{f'(x)} \right| = \frac{1}{|1+B|} \leq \sum_{k=0} |B|^k = \frac{1}{1-|B|},$$

provided $|B| < 1$. Therefore we estimate $|B|$:

$$|B| \leq \sum_{k=2}^{d} k u^{k-1} = \sum_{k=1}^{d} k u^{k-1} - 1 \leq \sum_{k=1}^{\infty} k u^{k-1} - 1 = \frac{1}{(1-u)^2} - 1.$$

To verify whether $|B| < 1$ we solve $\frac{1}{(1-u)^2} - 1 = 1$.
The smallest positive root of $1 - 4u + 2u^2$ is $1 - \frac{\sqrt{2}}{2}$,
so $u < 1 - \frac{\sqrt{2}}{2} \Rightarrow |B| < 1$. Thus:

$$\left| \frac{f'(\zeta)}{f'(x)} \right| \leq \frac{1}{1 - \left(\frac{1}{(1-u)^2} - 1 \right)} = \frac{(1-u)^2}{1 - 4u + 2u^2}$$

estimating $f'(\zeta)/f'(x)$

$$\left|\frac{f'(\zeta)}{f'(x)}\right| = \frac{1}{|1+B|} \leq \sum_{k=0} |B|^k = \frac{1}{1-|B|},$$

provided $|B| < 1$. Therefore we estimate $|B|$:

$$|B| \leq \sum_{k=2}^{d} k u^{k-1} = \sum_{k=1}^{d} k u^{k-1} - 1 \leq \sum_{k=1}^{\infty} k u^{k-1} - 1 = \frac{1}{(1-u)^2} - 1.$$

To verify whether $|B| < 1$ we solve $\frac{1}{(1-u)^2} - 1 = 1$.
The smallest positive root of $1 - 4u + 2u^2$ is $1 - \frac{\sqrt{2}}{2}$,
so $u < 1 - \frac{\sqrt{2}}{2} \Rightarrow |B| < 1$. Thus:

$$\left|\frac{f'(\zeta)}{f'(x)}\right| \leq \frac{1}{1 - \left(\frac{1}{(1-u)^2} - 1\right)} = \frac{(1-u)^2}{1 - 4u + 2u^2}$$

Proof of Theorem 1

Theorem 1

Assume $f(\zeta) = 0$ and $f'(\zeta) \neq 0$.

If $|x - \zeta| \leq \dfrac{3-\sqrt{7}}{2\gamma(f,\zeta)}$, then x is an approximate zero for ζ.

Proof. For x to be an approximate zero of f for the root ζ we must have $|N_f(x) - \zeta| \leq \frac{1}{2}|x - \zeta|$.

Applying Lemma 3, we consider $\dfrac{u}{1 - 4u + 2u^2} = \dfrac{1}{2}$.

This equation has two positive solutions: $(3 \pm \sqrt{7})/2$.

We take the smallest solution as a bound for

$$u = \gamma(f,\zeta)|x - \zeta| \leq \frac{3-\sqrt{7}}{2},$$

which leads to the criterion $|x - \zeta| \leq (3 - \sqrt{7})/(2\gamma(f,\zeta))$. □

Proof of Theorem 1

Theorem 1

Assume $f(\zeta) = 0$ and $f'(\zeta) \neq 0$.

If $|x - \zeta| \leq \dfrac{3 - \sqrt{7}}{2\gamma(f, \zeta)}$, then x is an approximate zero for ζ.

Proof. For x to be an approximate zero of f for the root ζ we must have $|N_f(x) - \zeta| \leq \frac{1}{2}|x - \zeta|$.

Applying Lemma 3, we consider $\dfrac{u}{1 - 4u + 2u^2} = \dfrac{1}{2}$.

This equation has two positive solutions: $(3 \pm \sqrt{7})/2$.

We take the smallest solution as a bound for

$$u = \gamma(f, \zeta)|x - \zeta| \leq \frac{3 - \sqrt{7}}{2},$$

which leads to the criterion $|x - \zeta| \leq (3 - \sqrt{7})/(2\gamma(f, \zeta))$. □

Proof of Theorem 1

Theorem 1

Assume $f(\zeta) = 0$ and $f'(\zeta) \neq 0$.

If $|x - \zeta| \leq \dfrac{3-\sqrt{7}}{2\gamma(f,\zeta)}$, then x is an approximate zero for ζ.

Proof. For x to be an approximate zero of f for the root ζ we must have $|N_f(x) - \zeta| \leq \frac{1}{2}|x - \zeta|$.

Applying Lemma 3, we consider $\dfrac{u}{1 - 4u + 2u^2} = \dfrac{1}{2}$.

This equation has two positive solutions: $(3 \pm \sqrt{7})/2$.

We take the smallest solution as a bound for

$$u = \gamma(f,\zeta)|x - \zeta| \leq \frac{3-\sqrt{7}}{2},$$

which leads to the criterion $|x - \zeta| \leq (3 - \sqrt{7})/(2\gamma(f,\zeta))$. □

Proof of Theorem 1

Theorem 1

Assume $f(\zeta) = 0$ and $f'(\zeta) \neq 0$.

If $|x - \zeta| \leq \dfrac{3 - \sqrt{7}}{2\gamma(f,\zeta)}$, then x is an approximate zero for ζ.

Proof. For x to be an approximate zero of f for the root ζ we must have $|N_f(x) - \zeta| \leq \frac{1}{2}|x - \zeta|$.

Applying Lemma 3, we consider $\dfrac{u}{1 - 4u + 2u^2} = \dfrac{1}{2}$.

This equation has two positive solutions: $(3 \pm \sqrt{7})/2$.

We take the smallest solution as a bound for

$$u = \gamma(f,\zeta)|x - \zeta| \leq \frac{3 - \sqrt{7}}{2},$$

which leads to the criterion $|x - \zeta| \leq (3 - \sqrt{7})/(2\gamma(f,\zeta))$. □

Alpha Theory to Certify Roots

Newton's method in $\mathbb{P}^n$

Consider $f(\mathbf{x}) = \mathbf{0}$ with $\mathbf{x} = [x_0 : x_1 : x_2 : \cdots : x_n] \in \mathbb{P}^n$.

We have n equations in $n+1$ variables. For uniqueness, restrict Newton's method to the tangent space $T_\mathbf{x}$ at $\mathbf{x}$:

$$T_\mathbf{x} = \{ \, \mathbf{y} \in \mathbb{P}^{n+1} \mid x_0 y_0 + x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = 0 \, \}.$$

Denote the inverse of the Jacobian matrix J_f of f restricted to $T_\mathbf{x}$ by $J_f(\mathbf{x})|_{T_\mathbf{x}}^{-1}$.

The Newton operator in $\mathbb{P}^n$ is $N_f(\mathbf{x}) = \mathbf{x} - J_f(\mathbf{x})|_{T_\mathbf{x}}^{-1} f(\mathbf{x})$.

an example

Consider

$$f(x_1, x_2) = \begin{cases} x_1 x_2 - 1 = 0 \\ x_1^2 - 0.002 = 0 \end{cases}$$

where smaller values for `0.002` lead to larger values for x_2.

In $\mathbb{P}^2$: we have $h_1(\mathbf{z}) = z_1 z_2 - z_0^2$ and $h_2(\mathbf{z}) = z_1^2 - 0.002 z_0^2$.

The linear system we solve to compute the update $\Delta\mathbf{z}$ is

$$\begin{bmatrix} -2z_0 & z_2 & z_1 \\ -0.004 z_0 & 2z_1 & 0 \\ z_0 & z_1 & z_2 \end{bmatrix} \Delta\mathbf{z} = \begin{bmatrix} h_1(\mathbf{z}) \\ h_2(\mathbf{z}) \\ 0 \end{bmatrix}.$$

For this small example, compute coordinates for $\mathbf{z}$ as

- $x_1 =$ `sqrt(0.002)` and $x_2 =$ `1/sqrt(0.002)`;
- take for $z_0 = 1/\max(x_1, x_2)$.

With higher working precision, $\mathbf{z} + \Delta\mathbf{z}$ will be more accurate.

an example

Consider

$$f(x_1, x_2) = \begin{cases} x_1 x_2 - 1 = 0 \\ x_1^2 - 0.002 = 0 \end{cases}$$

where smaller values for `0.002` lead to larger values for x_2.
In $\mathbb{P}^2$: we have $h_1(\mathbf{z}) = z_1 z_2 - z_0^2$ and $h_2(\mathbf{z}) = z_1^2 - 0.002 z_0^2$.
The linear system we solve to compute the update $\Delta\mathbf{z}$ is

$$\begin{bmatrix} -2z_0 & z_2 & z_1 \\ -0.004z_0 & 2z_1 & 0 \\ z_0 & z_1 & z_2 \end{bmatrix} \Delta\mathbf{z} = \begin{bmatrix} h_1(\mathbf{z}) \\ h_2(\mathbf{z}) \\ 0 \end{bmatrix}.$$

For this small example, compute coordinates for $\mathbf{z}$ as

- $x_1 =$ `sqrt(0.002)` and $x_2 =$ `1/sqrt(0.002)`;
- take for $z_0 = 1/\max(x_1, x_2)$.

With higher working precision, $\mathbf{z} + \Delta\mathbf{z}$ will be more accurate.

an example

Consider

$$f(x_1, x_2) = \begin{cases} x_1 x_2 - 1 = 0 \\ x_1^2 - 0.002 = 0 \end{cases}$$

where smaller values for `0.002` lead to larger values for x_2.

In $\mathbb{P}^2$: we have $h_1(\mathbf{z}) = z_1 z_2 - z_0^2$ and $h_2(\mathbf{z}) = z_1^2 - 0.002 z_0^2$.

The linear system we solve to compute the update $\Delta\mathbf{z}$ is

$$\begin{bmatrix} -2z_0 & z_2 & z_1 \\ -0.004 z_0 & 2z_1 & 0 \\ z_0 & z_1 & z_2 \end{bmatrix} \Delta\mathbf{z} = \begin{bmatrix} h_1(\mathbf{z}) \\ h_2(\mathbf{z}) \\ 0 \end{bmatrix}.$$

For this small example, compute coordinates for $\mathbf{z}$ as

- $\mathbf{x}_1 =$ `sqrt(0.002)` and $x_2 =$ `1/sqrt(0.002)`;
- take for $z_0 = 1/\max(x_1, x_2)$.

With higher working precision, $\mathbf{z} + \Delta\mathbf{z}$ will be more accurate.

generalizing $\gamma(f, x)$

For systems, we define

$$\begin{aligned} \gamma \ : \ \mathbb{P}[x] \times \mathbb{P}^n \rightarrow & \quad \mathbb{R} \\ (f, x) \quad \mapsto \gamma(f, x) = & \ ||\mathbf{x}|| \max_{k=2}^{d} \left|\left| J_f(\mathbf{x})|_{T_\mathbf{x}}^{-1} \frac{D^k f(x)}{k!} \right|\right|^{1/(k-1)} \end{aligned}$$

where d is the largest degree of the polynomials in f, and $D^k f$ is the kth derivative of f, considered as a k-linear map, $D^1 f = J_f$. With $|| \cdot ||$ we denote vector and matrix norms.

a criterion

Theorem 3

Let $f(\mathbf{x}) = \mathbf{0}$ be a polynomial system with coordinates in projective space and $\zeta \in \mathbb{P}^n$ a root.
If

$$\frac{||z - \zeta||}{||\zeta||}\gamma(f, \zeta) \leq \frac{3 - \sqrt{7}}{2},$$

then z is an approximate zero of f for ζ.

Summary + Exercises

We looked at the region of convergence of Newton's method which led to the notion of approximate roots.

Exercises:

1. Consider formula $|x_k - \zeta| \leq \left(\frac{1}{2}\right)^{2^k-1} |z - \zeta|$ for an approximate zero z close enough to a zero ζ of f. Assuming $|z - \zeta| \leq 0.1$ and given some $\epsilon > 0$, derive a bound on k, the number of iterations so that $|x_k - \zeta| \leq \epsilon$. Illustrate your bound with numerical examples.
2. Define a Maple function to evaluate γ for any polynomial f at any point x.
   Instead of Maple you may also use another computer algebra system.

more exercises

3. Consider the polynomial $f(x) = x(x - r)$, where r is a parameter for the second root of f.
   1. For r = 1, compute the radius $\frac{3-\sqrt{7}}{2\gamma(f,\zeta)}$ for all roots ζ.
   2. Let r go to zero (take $r = 1/10^k$, for $k = 1, 2, \ldots$) and recompute $\frac{3-\sqrt{7}}{2\gamma(f,\zeta)}$ for all the roots.
      What do you observe as the second root gets closer to zero?
   3. Let r grow larger (take $r = 10^k$, for $k = 1, 2, \ldots$) and recompute $\frac{3-\sqrt{7}}{2\gamma(f,\zeta)}$ for all the roots.
      What do you observe as the second root moves farther away?

and more exercises

4. Suppose that ζ_1 and ζ_2 are two regular roots of f, i.e.: $f(\zeta_1) = 0 = f(\zeta_2)$ and $f'(\zeta_1) \neq 0 \neq f'(\zeta_2)$. Show that ζ_1 and ζ_2 are separated as follows:

$$|\zeta_1 - \zeta_2| \geq \frac{5 - \sqrt{17}}{4\gamma(f, \zeta_2)}.$$

5. Use a computer algebra package to write a procedure to perform one step of the projective Newton method. On input are the polynomials in the system $f(\mathbf{x}) = \mathbf{0}$ with an approximate solution in the original affine coordinates. The procedure performs the transformation to projective coordinates $\mathbf{z}$, computes the extended Jacobian matrix and solves a linear system for $\Delta\mathbf{z}$. On return is the updated solution in projective and affine coordinates.