Binomial Systems

1. Binomial Ideals
   - definition and properties
   - solving a zero dimensional pure difference ideal

2. Commuting Birth-and-Death Processes
   - models from ecology and queuing theory
   - a system of quadratic polynomials

3. Cellular decompositions
   - decomposing binomial ideals

4. Using Macaulay2
   - running examples with the package Binomials
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Consider $K[x]$, with $K^* = K \setminus \{0\}$.

Typically we will assume that $K$ is algebraically closed, so $K = \mathbb{C}$ is our default coefficient field. Then

$$ I = \langle c_a x^a - c_b x^b \mid a, b \in \mathbb{N}^n, c_a, c_b \in K^* \rangle $$

is a binomial ideal. A polynomial is a binomial if it has exactly two monomials with a nonzero coefficient.

A binomial ideal is generated by binomials.

**Definition**

A *pure difference ideal* is an ideal generated by differences of monic monomials, i.e.: all generators are of the form $x^a - x^b$. 
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Proposition

Let $I$ be a zero dimensional pure difference ideal. There is a primitive root of unity $\xi$, such that all complex solutions of $I$ are contained in the cyclotomic field $\mathbb{Q}(\xi)$.

Proof. Let $\mathcal{G}$ be a lexicographic Gröbner basis.

- Because all $S$-polynomials are pure difference binomials, $\mathcal{G}$ consists of pure difference binomials.

- As the ideal is zero dimensional and because a lexicographic order eliminates, at least one of the binomials in $\mathcal{G}$ is univariate.

- The solutions of the univariable equations exists in a cyclotomic field. By substituting the solution for that variable in the other equations, an univariate equation in another variable is obtained.

- After extending the partial solutions, all roots of unity encountered during univariate solving define $\mathbb{Q}(\xi)$ where the solutions live. \qed
toric varieties

Because the exponents determine the structure of the ideal, we then define a toric ideal as

\[ I_A = \langle x^u - x^v \mid u, v \in \mathbb{N}^n \text{ and } A u = A v \rangle. \]

The solution set of a toric ideal is a toric variety.

As an alternative to the ideal description, a toric variety over \( \mathbb{C} \) is defined as

- a complex algebraic variety with an action of \( (\mathbb{C}^\ast)^n \) and
- a dense open subset isomorphic to \( (\mathbb{C}^\ast)^n \) carrying the regular action.

That is: a toric variety is an algebraic torus closure.

In polyhedral homotopies: at \( \infty \) and at 0 are equivalent.
Binomial primary decompositions

Binomial ideals have special properties, for instance:

**Theorem (Theorem 2.6 in [Eisenbud-Sturmfels, 1996])**

\( K \) is algebraically closed and \( I \) is a binomial ideal in \( K[x] \), then every associated prime of \( I \) is generated by binomials.

The condition that \( K \) is algebraically closed is essential:

over \( \mathbb{Q} \): \( \langle x^3 - 1 \rangle = \langle x - 1 \rangle \cap \langle x^2 + x + 1 \rangle \).

If we extend \( \mathbb{Q} \) with \( w = e^{(2\pi \sqrt{-1})/3} \), then over \( \mathbb{Q}(w) \):

\( \langle x^3 - 1 \rangle = \langle x - 1 \rangle \cap \langle x + (1 - \sqrt{-3})/2 \rangle \cap \langle x + (1 + \sqrt{-3})/2 \rangle \).
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Commuting Birth-and-Death Processes

An application of combinatorics and algebraic statistics:

- Models arising in ecology and queuing theory study population sizes and numbers of individual waiting in a queue.

- Markov chains are described by tridiagonal transition matrices $P$, $P(i, j)$ is the probability of going from step $i$ to $j$.

- In a higher-dimensional model the state space is a product of intervals in higher-dimensional lattices, e.g.:
  - ecology: keep track of the type of individuals in a population;
  - queuing: several servers have each their own set of customers.

- The mathematical tools are
  - one dimension: orthogonal polynomials;
  - higher dimension: binomial primary decomposition.
an example in dimension two

Define a grid \( E = \{0, 1, \ldots, m\} \times \{0, 1, \ldots, n\} \)
where \((i, j)\) is connected to \((k, \ell)\) \iff \(|i - k| + |j - \ell| = 1\).

The transition probabilities are

- **go left:** \( L_{i,j} = \text{prob}\{Z_{k+1} = (i - 1, j) \mid Z_k = (i, j)\} \)
- **go right:** \( R_{i,j} = \text{prob}\{Z_{k+1} = (i + 1, j) \mid Z_k = (i, j)\} \)
- **go down:** \( D_{i,j} = \text{prob}\{Z_{k+1} = (i, j - 1) \mid Z_k = (i, j)\} \)
- **go up:** \( U_{i,j} = \text{prob}\{Z_{k+1} = (i, j + 1) \mid Z_k = (i, j)\} \)

Commuting relations:

\[
\begin{align*}
U_{i,j} R_{i,j+1} &= R_{i,j} U_{i+1,j} & \text{(up-right)} \\
D_{i,j+1} R_{i,j} &= R_{i,j+1} D_{i+1,j+1} & \text{(down-right)} \\
D_{i+1,j+1} L_{i+1,j} &= L_{i+1,j+1} D_{i,j+1} & \text{(down-left)} \\
U_{i+1,j} L_{i+1,j+1} &= L_{i+1,j} U_{i,j} & \text{(up-left)}
\end{align*}
\]
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a system of quadratic polynomials

\[ E = \{0, 1, \ldots, n_1\} \times \{0, 1, \ldots, n_2\} \times \ldots \times \{0, 1, \ldots, n_m\} \]

For all pairs \((i, j)\): \(1 \leq i < j \leq m\), the commuting requirement

\[
P(u, u + e_i)P(u + e_i, u + e_i + e_j) - P(u, u + e_j)P(u + e_j, u + e_i + e_j),
\]
\[
P(u, u + e_i)P(u + e_i, u + e_i - e_j) - P(u, u - e_j)P(u + e_j, u + e_i - e_j),
\]
\[
P(u, u - e_i)P(u - e_i, u - e_i + e_j) - P(u, u + e_j)P(u + e_j, u - e_i + e_j),
\]
\[
P(u, u - e_i)P(u - e_i, u - e_i - e_j) - P(u, u - e_j)P(u + e_j, u - e_i - e_j)
\]

is a system of quadratic polynomials in the unknowns \(P(u, v)\).
the ideal of commuting birth-and-death processes

Denote by $I^{(n_1,n_2,\ldots,n_m)}$ the ideal generated by the quadratic polynomials in the commuting requirement.

In the two dimensional case, $I^{(m,n)}$ is generated by $4mn$ quadratic binomials, for $(i,j): 0 \leq i < m$ and $0 \leq j < n$:

\[
\begin{align*}
U_{i,j}R_{i,j+1} - R_{i,j}U_{i+1,j} &= 0, \\
R_{i,j+1}D_{i+1,j+1} - D_{i,j+1}R_{i,j} &= 0, \\
D_{i+1,j+1}L_{i+1,j} - L_{i+1,j+1}D_{i,j+1} &= 0, \\
L_{i+1,j}U_{i,j} - U_{i+1,j}L_{i+1,j+1} &= 0.
\end{align*}
\]
the smallest example

The possibilities that

\[
\begin{pmatrix}
0 & 0 & R_{0,0} & 0 \\
0 & 0 & 0 & R_{0,1} \\
L_{1,0} & 0 & 0 & 0 \\
0 & L_{1,1} & 0 & 0
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
0 & U_{0,0} & 0 & 0 \\
D_{0,1} & 0 & 0 & 0 \\
0 & 0 & 0 & U_{1,0} \\
0 & 0 & D_{1,1} & 0
\end{pmatrix}
\]

commute are revealed by the primary decomposition of

\[
I^{(1,1)} = \langle U_{0,0} R_{0,1} - R_{0,0} U_{1,0}, R_{0,1} D_{1,1} - D_{0,1} R_{0,0}, D_{1,1} L_{1,0} - L_{1,1} D_{0,1}, L_{1,0} U_{0,0} - U_{1,0} L_{1,1} \rangle.
\]
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Consider $\mathcal{E} \subseteq \{1, 2, \ldots, n\}$ and denote the algebraic torus corresponding to $\mathcal{E}$ by

$$(\mathbb{K}^*)^\mathcal{E} = \{ \mathbf{x} \in \mathbb{K}^n | x_i \neq 0 \text{ for } i \in \mathcal{E} \text{ and } x_j = 0 \text{ for } j \notin \mathcal{E} \}.$$ 

The central definition is

**Definition**

A proper binomial ideal $I$ in $\mathbb{K}[\mathbf{x}]$ is *cellular* if each variable $x_i$ is either a nonzerodivisor or nilpotent modulo $I$.

Primary ideals $I$ are cellular as every element in $\mathbb{K}[\mathbf{x}]/I$ is either nilpotent or a nonzerodivisor.
characterizing cellular ideals

We have a characterization for an ideal $I$ being cellular in the following lemma.

**Lemma**

A proper binomial ideal $I$ in $\mathbb{K}[x]$ is cellular if and only if there exists a set $\mathcal{E} \subseteq \{1, 2, \ldots, n\}$ of indices of variables in $x$ such that

1. $I = \left( I : \left( \prod_{i \in \mathcal{E}} x_i \right)^\infty \right)$; and

2. For every $i \notin \mathcal{E}$, there exists an integer $d_i \geq 0$ such that $\langle x_i^{d_i} \mid i \notin \mathcal{E} \rangle$ is contained in $I$. 
The Binomials package in Macaulay 2 provides an implementation of the following recursive algorithm:

**Algorithm [cellular decomposition]**

Input: a binomial ideal $I$.
Output: a cellular decomposition of $I$.

1. If $I$ is cellular, then return $I$.
2. Choose $x_i$ that is a zerodivisor but not nilpotent modulo $I$.
3. Determine the power $m$ such that $(I : x_i^m) = (I : x_i^\infty)$.
4. Call the algorithm on $(I : x_i^m)$ and $I + \langle x_i^m \rangle$. 
solving toric ideals

Input: a zero dimensional toric ideal \( I \).
Output: roots of unity to extend \( \mathbb{Q} \) and solutions in \( V(I) \).

1. Compute a cellular decomposition of \( I \).
2. For each cellular component do
   2.1 Set the noncell variables to zero and determine
       the product \( D := \prod_{i \notin \mathcal{E}} d_i \)
       of the minimal powers of the noncell variables.
   2.2 Compute a lexicographic Gröbner basis and
       solve the lattice ideal of the cellular component,
       adjoining roots of unity.
   2.3. Save each solution \( D \) times.
3. Compute the least common multiple \( m \) of the powers
   of the adjoined roots of unity
   and construct the cyclotomic field \( \mathbb{Q}(w_m) \).
4. Return the list of solutions as elements in \( \mathbb{Q}(w_m) \).
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running Binomials

The package Binomials of Thomas Kahle is in Macaulay2.

i1 : S = QQ[x,y,z];
i2 : I = ideal(x^2-y,y^3-z,x*y-z);
i3 : loadPackage "Binomials";
i4 : binomialSolve I
BinomialSolve created a cyclotomic field of order 3

o4 = {{1, 1, 1}, {- \(ww \) - 1, \(ww\) , 1},
    3 3
    \(ww\), - \(ww\) - 1, 1},
    3 3
    {0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
i5 : degree I
o5 = 6
binomial primary decomposition

i6 : BPD I
Running cellular decomposition:
cellular components found: 1
cellular components found: 2
Decomposing cellular components:
Decomposing cellular component: 1 of 2
1 monomial to consider for this cellular component
BinomialSolve created a cyclotomic field of order 3
done
Decomposing cellular component: 2 of 2
3 monomials to consider for this cellular component
done
Removing redundant components...
4 Ideals to check
3 Ideals to check
2 Ideals to check
1 Ideals to check
0 redundant ideals removed.
Computing mingens of result.
The primary decomposition of $\langle x^2 - y, y^3 - z, xy - z \rangle$ is

$$o6 = \{ \text{ideal} \ (z - 1, y - 1, x - 1),$$

$$\text{ideal} \ (z - 1, y - \text{ww}, x + \text{ww} + 1), \quad \text{ideal} \ (z - 1, y + \text{ww} + 1, x - \text{ww}),$$

$$\text{ideal} \ (z, y, x*y, x - y) \}$$
We consider the last ideal in the primary decomposition

```
i7 : I = ideal(z,y^2,x*y,x^2 - y);
i8 : binomialAssociatedPrimes I
3 monomials to consider for this cellular component

o8 = {ideal (z, y, x)}
```
cellular decompositions

```plaintext
i2 : S = QQ[x1,x2,x3,x4,x5];
i3 : I = ideal(x1*x4^2-x2*x5^2, x1^3*x3^3-x2^4*x4^2, x2*x4^8-x3^3*x5^6);
i4 : I
   2  2  3  3  4  2
o4 = ideal (x1 x4 - x2 x5 , x1 x3 - x2 x4 ,
             8  3  6
     x2 x4 - x3 x5 )

i5 : BCD I
```
cellular components found: 1
redundant component
redundant component

cellular components found: 2

\[
\begin{array}{cccccc}
2 & 2 & 3 & 3 & 4 & 2 \\
\end{array}
\]

\[
o_5 = \{ \text{ideal} \ (x_1 \cdot x_4 - x_2 \cdot x_5, \ x_1 \cdot x_3 - x_2 \cdot x_4, \\
3 & 4 & 2 & 3 & 2 \\
x_2 \cdot x_4 - x_1 \cdot x_3 \cdot x_5, \\
2 & 6 & 3 & 4 & 2 & 3 & 2 \\
x_2 \cdot x_4 - x_2 \cdot x_4 - x_1 \cdot x_3 \cdot x_5, \\
2 & 6 & 3 & 4 & 8 & 3 & 6 \\
x_2 \cdot x_4 - x_1 \cdot x_3 \cdot x_5, \ x_2 \cdot x_4 - x_3 \cdot x_5, \\
2 & 2 & 2 & 5 \\
\text{ideal} \ (x_1, \ x_1 \cdot x_4 - x_2 \cdot x_5, \ x_2, \\
6 & 4 & 2 & 8 \\
x_5, \ x_2 \cdot x_4, \ x_4) \}
\]
Summary + Exercises

Binomial ideals are an interesting class of problems.

Exercises:

1. Explore the package Binomials in Macaulay2.
2. Explore the capabilities in CoCoA for handling binomial ideals.
3. Explore the capabilities in Sage/Singular for binomial ideals.