Elimination Methods

MCS 563 Lecture 2
Analytic Symbolic Computation
Jan Verschelde, 12 January 2011

Elimination Methods

intersecting two circles

Consider the unit circle, intersected with a general circle centered at $(c, 0)$ and with radius r:

$$f(x_1, x_2) = \left\{ \begin{array}{r} x_1^2 + x_2^2 - 1 = 0 \\ (x_1 - c)^2 + x_2^2 - r^2 = 0. \end{array} \right.$$

Subtracting the second equation from the first: $2cx_1 - c^2 - 1 + r^2 = 0$, at most two solutions.

According to Bézout's theorem:
two quadrics can have up to four isolated solutions.

Similar as: two parallel lines meet at infinity, the system of two intersecting circles has solutions at infinity.

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a projective transformation

We embed the system into projective space, the map ψ

$$\psi : \mathbb{C}^n \rightarrow \mathbb{P}^n : (x_1, x_2, \ldots, x_n) \mapsto [x_0 : x_1 : x_2 : \cdots : x_n]$$

defines an embedding of $\mathbb{C}^n$ into projective space $\mathbb{P}^n$, with

$$[x_0 : x_1 : x_2 : \cdots : x_n] = \left[1 : \frac{x_1}{x_0} : \frac{x_2}{x_0} : \cdots : \frac{x_n}{x_0} \right].$$

Points are mapped to equivalence classes, defined by

$$\begin{aligned} [x_0 : x_1 : \cdots : x_n] & \sim [y_0 : y_1 : \cdots : y_n] \\ & \Leftrightarrow \exists \lambda \neq 0 : x_i = \lambda y_i, i = 0, 1, \ldots, n. \end{aligned}$$

If $x_0 \rightarrow 0$, then $x_i \rightarrow \infty$, $i = 1, 2, \ldots, n$.

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Elimination Methods

1. Projective Coordinates
   - intersecting two circles
   - computing solutions at infinity
2. Molecular Configurations
   - transformation to half angles
3. Resultants
   - the Sylvester matrix
4. Cascading Resultants
   - computing discriminants
5. The Main Theorem of Elimination Theory

computing solutions at infinity

To find solutions at infinity of

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we replace x_1 by x_1/x_0 and x_2 by x_2/x_0, to obtain:

$$f([x_0 : x_1 : x_2]) = \left\{ \begin{array}{r} x_1^2 + x_2^2 - x_0^2 = 0 \\ x_1^2 - cx_1x_0 + x_2^2 - r^2x_0^2 = 0. \end{array} \right.$$

The system is homogeneous: all monomials have degree 2.

Solve $f([0 : x_1 : x_2]) = 0$ to find $[0 : 1 : \pm\sqrt{-1}]$.

Applications of projective coordinates:

- State Bézout's theorem for solutions in $\mathbb{P}^n$.
- Rephrase a numerically ill-conditioned problem.

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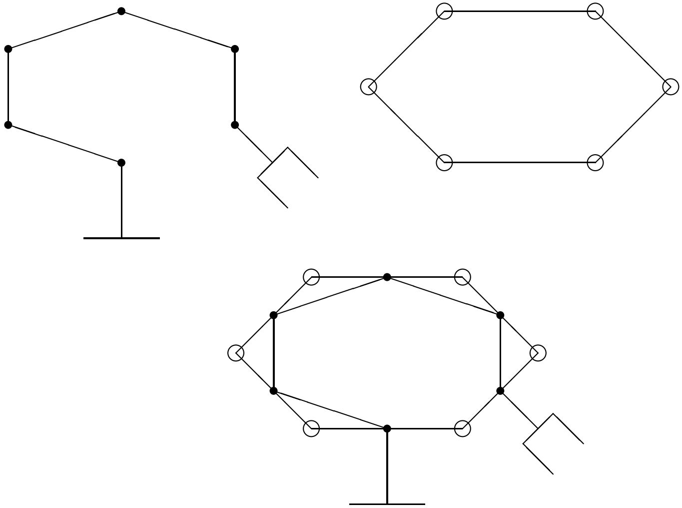
Molecular Configurations

A robot arm at the left and a configuration of 6 atoms:



a system of equations

Flap angles θ_i are angles between the triangle at the center and adjacent triangles (this is a spatial configuration). Keeping distances between the atoms fixed is expressed by

$$\left\{ \begin{array}{rcl} \alpha_{11} + \alpha_{12} \cos(\theta_2) + \alpha_{13} \cos(\theta_3) & & \\ + \alpha_{14} \cos(\theta_2) \cos(\theta_3) + \alpha_{15} \sin(\theta_2) \sin(\theta_3) & = & 0 \\ \alpha_{21} + \alpha_{22} \cos(\theta_3) + \alpha_{23} \cos(\theta_1) & & \\ + \alpha_{24} \cos(\theta_3) \cos(\theta_1) + \alpha_{25} \sin(\theta_3) \sin(\theta_1) & = & 0 \\ \alpha_{31} + \alpha_{32} \cos(\theta_1) + \alpha_{33} \cos(\theta_2) & & \\ + \alpha_{34} \cos(\theta_1) \cos(\theta_2) + \alpha_{35} \sin(\theta_1) \sin(\theta_2) & = & 0 \\ \cos^2(\theta_1) + \sin^2(\theta_1) - 1 & = & 0 \\ \cos^2(\theta_2) + \sin^2(\theta_2) - 1 & = & 0 \\ \cos^2(\theta_3) + \sin^2(\theta_3) - 1 & = & 0 \end{array} \right.$$

Elimination Methods

transformation to half angles

Letting $t_i = \tan\left(\frac{\theta_i}{2}\right)$, for $i = 1, 2, 3$, implies

$$\cos(\theta_i) = \frac{1 - t_i^2}{1 + t_i^2} \quad \text{and} \quad \sin(\theta_i) = \frac{2t_i}{1 + t_i^2}.$$

After clearing denominators:

$$\left\{ \begin{array}{l} \beta_{11} + \beta_{12} t_2^2 + \beta_{13} t_3^2 + \beta_{14} t_2 t_3 + \beta_{15} t_2^2 t_3^2 = 0 \\ \beta_{21} + \beta_{22} t_3^2 + \beta_{23} t_1^2 + \beta_{24} t_3 t_1 + \beta_{25} t_3^2 t_1^2 = 0 \\ \beta_{31} + \beta_{32} t_1^2 + \beta_{33} t_2^2 + \beta_{34} t_1 t_2 + \beta_{35} t_1^2 t_2^2 = 0 \end{array} \right.$$

where the β_{ij} are the input coefficients.

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a lemma

Lemma

Two polynomials f and g have a common factor $\Leftrightarrow$ there exist two nonzero polynomials A and B such that $Af + Bg = 0$, with $\deg(A) < \deg(g)$ and $\deg(B) < \deg(f)$.

Proof.

$\Rightarrow$ Let $f = f_1 h$ and $g = g_1 h$, then $A = -g_1$ and $B = -f_1$:
$Af + Bg = -g_1 f_1 h + f_1 g_1 h = 0$.

$\Leftarrow$ Suppose f and g have no common factor.
By the Euclidean algorithm we then have that
$\mathrm{GCD}(f, g) = 1 = \widetilde{A}f + \widetilde{B}g$. Assume $B \neq 0$.
Then $B = 1B = (\widetilde{A}f + \widetilde{B}g)B$. Using $Bg = -Af$ we obtain
$B = (\widetilde{A}B - \widetilde{B}A)f$. As $B \neq 0$, it means that $\deg(B) \geq \deg(f)$
which gives a contradiction.
Thus f and g must have a common factor. $\square$

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the Sylvester matrix

$A(x)f(x) + B(x)g(x) = 0$,
$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$, $A(x) = c_0 + c_1x$,
$g(x) = b_0 + b_1x + b_2x^2$, and $B(x) = d_0 + d_1x + d_2x^2$.

$$\left\{ \begin{array}{ccccccccccccc}
x^4 & : & a_3c_1 & & & + & b_2d_2 & & & & & = & 0 \\
x^3 & : & a_2c_1 & + & a_3c_0 & + & b_1d_2 & + & b_2d_1 & & & = & 0 \\
x^2 & : & a_1c_1 & + & a_2c_0 & + & b_0d_2 & + & b_1d_1 & + & b_2d_0 & = & 0 \\
x^1 & : & a_0c_1 & + & a_1c_0 & & & & b_0d_1 & + & b_1d_0 & = & 0 \\
x^0 & : & & & a_0c_0 & & & & & & b_0d_0 & = & 0.
\end{array} \right.$$

In matrix form:

$$\left[\begin{array}{ccccc}
a_3 & & b_2 & & \\
a_2 & a_3 & b_1 & b_2 & \\
a_1 & a_2 & b_0 & b_1 & b_2 \\
a_0 & a_1 & & b_0 & b_1 \\
& a_0 & & & b_0
\end{array} \right]
\left[\begin{array}{c} c_1 \\ c_0 \\ d_2 \\ d_1 \\ d_0 \end{array} \right]
=
\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right].$$

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In matrix form:

$$\begin{bmatrix} a_3 & & b_2 & & \\ a_2 & a_3 & b_1 & b_2 & \\ a_1 & a_2 & b_0 & b_1 & b_2 \\ a_0 & a_1 & & b_0 & b_1 \\ & a_0 & & & b_0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_0 \\ d_2 \\ d_1 \\ d_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

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x^1 & : & a_0c_1 & + & a_1c_0 & & & & b_0d_1 & + & b_1d_0 & = & 0 \\
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In matrix form:

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\end{bmatrix}
\begin{bmatrix} c_1 \\ c_0 \\ d_2 \\ d_1 \\ d_0 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

the resultant

The matrix of the linear system $A(x)f(x) + B(x)g(x) = 0$ is the Sylvester matrix. Its determinant is the resultant.

The resultant is the condition on the coefficients of f and g for f and g to have a common factor.

In Maple:

```
[> sm := LinearAlgebra[SylvesterMatrix](f,g,x);
[> rs := LinearAlgebra[Determinant](sm);
```

For several variables: eliminate one variable by hiding the other variables in the coefficients.

Geometric interpretation: elimination $=$ projection.

For approximate coefficients, the determinant is replaced by numerical rank revealing methods via singular value decomposition or via a QR decomposition.

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the discriminant

The system

$$f(x,y) = \left\{ \begin{array}{r} x^2 + y^2 - 1 = 0 \\ (x-c)^2 + y^2 - r^2 = 0 \end{array} \right.$$

has exactly two solutions

$$\left[x = \frac{-r^2 + c^2 + 1}{2c}, y = \pm \frac{\sqrt{-r^4 + 2c^2r^2 + 2r^2 - c^4 + 2c^2 - 1}}{2c} \right]$$

except for those c and r satisfying

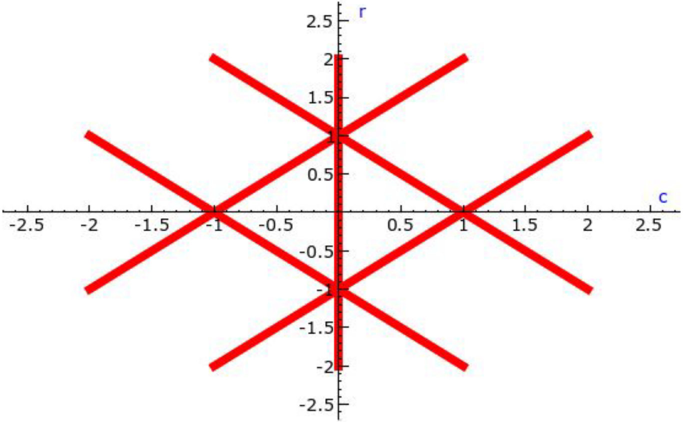
$$D(c,r) = 256(c-r-1)^2(c-r+1)^2(c+r-1)^2(c+r+1)^2c^4 = 0.$$

The polynomial $D(c,r)$ is the discriminant for the system.

the discriminant variety

The *discriminant variety* is the solution set of the discriminant.

$$D(c,r) = 256(c-r-1)^2(c-r+1)^2(c+r-1)^2(c+r+1)^2c^4 = 0.$$



computing discriminants

Given a system f with parameters, perform 2 steps:

1. Let E be $(f, \det(J_f))$, where J_f is the Jacobian matrix.
2. Eliminate from E all the indeterminates.

What remains after elimination is the discriminant.

For the intersecting two circles problem, we solve

$$J_f = \begin{bmatrix} 2x & 2y \\ 2(x-c) & 2y \end{bmatrix} \qquad E(x,y,c,r) = \left\{ \begin{array}{r} x^2 + y^2 - 1 = 0 \\ (x-c)^2 + y^2 - r^2 = 0 \\ 4cy = 0 \end{array} \right.$$

$$det(J_f) = 4xy - 4(x-c)y$$

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using Sage

```
sage: x,y,c,r = var(’x,y,c,r’)
sage: f1 = x^2 + y^2 - 1
sage: f2 = (x-c)^2 + y^2 - r^2
sage: J = matrix([[diff(f1,x),diff(f1,y)],
                  [diff(f2,x),diff(f2,y)]])
sage: dJ = det(J)
sage: print f1, f2, dJ
```

```
     2    2
    y  + x  - 1
 2          2    2
y  + (x - c)  - r
4 x y - 4 (x - c) y
```

using Singular in Sage

We convert the polynomials to a ring:

```
sage: R.<x,y,c,r> = QQ[]
sage: E = [R(f1),R(f2),R(dJ)]
```

Cascading resultants, we first eliminate x and then y:

```
sage: e01 = singular.resultant(E[0],E[1],x)
sage: e12 = singular.resultant(E[1],E[2],x)
sage: dsc = singular.resultant(e01,e12,y)
sage: factor(R(dsc))
(256) * (c - r - 1)^2 * (c - r + 1)^2
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some elimination theory

Resultants provide a constructive proof for the following.

Theorem (the main theorem of elimination theory)

Let $V \subset \mathbb{P}^n$ be the solution set of a polynomial system. Then the projection of V onto $\mathbb{P}^{n-1}$ is again the solution set of a system of polynomial equations.

Projective coordinates are needed, e.g.: $x_1 x_2 - 1 = 0$.

Summary + Exercises

We defined projective coordinates and resultants to give a constructive proof of the main theorem of elimination theory.

Exercises:

1. For the system to intersect two circles, determine the exceptional values for the parameters c and r for which there are infinitely many solutions. Justify your answer.
2. Derive the formulas for the transformation which uses half angles, used to transform a sin/cos system into a polynomial system.
3. Consider $f(x) = ax^2 + bx + c$. Use resultant methods available in Maple or Sage to compute the discriminant of this polynomial.

more exercises

④ Use Maple or Sage to compute resultants to solve

$$\left\{ \begin{array}{c} x_1 x_2 - 1 = 0 \\ x_1^2 + x_2^2 - 1 = 0. \end{array} \right.$$

⑤ Consider the system

$$\left\{ \begin{array}{l} x^2 + y^2 - 2 = 0 \\ x^2 + \frac{y^2}{5} - 1 = 0. \end{array} \right.$$

Show geometrically by making a plot of the two curves using Maple or Sage that elimination will lead to polynomials with double roots. Compute resultants (as univariate polynomials in x or y) and show algebraically that they have double roots.

and more exercises

6. Consider as choice in system of molecular configurations for the β_{ij} coefficients the matrix

$$\begin{bmatrix} -13 & -1 & -1 & 24 & -1 \\ -13 & -1 & -1 & 24 & -1 \\ -13 & -1 & -1 & 24 & -1 \end{bmatrix}.$$

   Try to solve the system using these coefficients. How many solutions do you find?

7. Extend the discriminant computation for the ellipse $(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$ and the circle $(x - c)^2 + y^2 = r^2$. Interpret the result by appropriate projections of the four dimensional parameter space onto a plane.
8. Examine the intersection of three spheres. What are the components of its discriminant variety?