Homotopies and Continuation

MCS 563 Lecture 3
Analytic Symbolic Computation
Jan Verschelde, 14 January 2011

Homotopies and Continuation

solving two quadrics

Suppose we want to solve

$$f(x_1, x_2) = \begin{cases} x_1^2 + x_2 - 3 = 0 \\ x_1 + 0.125x_2^2 - 1.5 = 0. \end{cases}$$

We embed the system in a family of systems:

$$h(\mathbf{x}, t) = (1-t)\begin{pmatrix} x_1^2 - 1 \\ x_2^2 - 1 \end{pmatrix} + t\begin{pmatrix} x_1^2 + x_2 - 3 \\ x_1 + 0.125x_2^2 - 1.5 \end{pmatrix} = \mathbf{0}.$$

This family is called a homotopy. At $t = 0$: $h(\mathbf{x}, 0) = \mathbf{0}$ is the start system. At $t = 1$: $h(\mathbf{x}, 1) = \mathbf{0}$ is the target system.

The homotopies defines solution paths $\mathbf{x}(t)$: $h(\mathbf{x}(t), t) = \mathbf{0}$.

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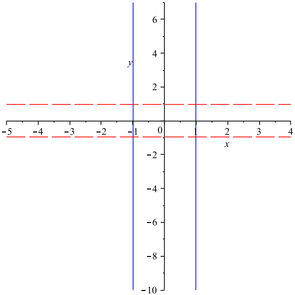
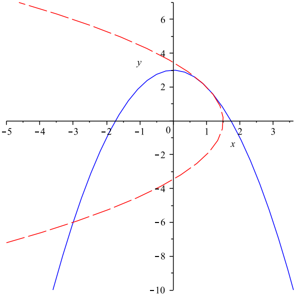
Homotopies and Continuation

deforming polynomials

We deform a pair of lines into two parabolas:



start system



target system

Going from target to start: we degenerate the two parabolas into a pair of lines (the method of degeneration).

using phc

phc is an executable, available at
`http://www.math.uic.edu/~jan/download.html`.

Make a file `example` with content:

```
2
  x^2 + y - 3;
  x + 0.125*y^2 - 1.5;
```

At the command prompt `$`, type

```
$ phc -b example

Do you want the output to file ? (y/n) n
```

the computed solutions

output (edited to fit the slide):

```
THE SOLUTIONS :
2 2
========================================
solution 1 :
t :  1.00000000000000E+00   0.00000000000000E+00
m : 1
the solution for t :
 x : -3.00000000000000E+00   0.00000000000000E+00
 y : -6.00000000000000E+00   0.00000000000000E+00
= err :  3.4E-16 = rco : 3.2E-01 = res : 4.4E-16 ==
solution 2 :
t :  1.00000000000000E+00   0.00000000000000E+00
m : 3
the solution for t :
 x :  1.00000000000000E+00  -7.33552924616013E-17
 y :  2.00000000000000E+00   1.48991504639649E-16
= err : 1.5E-15 = rco :  1.9E-19 = res : 1.1E-16 ==
```

a general start system

For any square system $f(\mathbf{x}) = \mathbf{0}$, we define

$$g(\mathbf{x}) = \begin{cases} x_1^{d_1} - 1 = 0, & d_1 = \deg(f_1) \\ x_2^{d_2} - 1 = 0, & d_2 = \deg(f_2) \\ \vdots & \vdots \\ x_n^{d_n} - 1 = 0, & d_n = \deg(f_n). \end{cases}$$

$\#\text{solutions} = d_1 \times d_2 \times \cdots \times d_n = D$ total degree.

Homotopies and Continuation

a problem of magnetism

Equations coming from a problem of magnetism in physics:

$$\left\{ \begin{array}{c} \displaystyle\sum_{i=-N}^{N} u(i)u(m-i) = u(m) \\ \displaystyle\sum_{i=-N}^{N} u(i) = 1 \end{array} \right.$$

with $m \in \{-N+1, -N, \ldots, N-1\}$,
$u(i) = u(-i)$, and $u(i) = 0$, for $|i| > N$.

#solutions is $2^N = D$ complex isolated solutions, but only solutions with all components in $[0, 1]$ matter.

Number of really interesting solutions equals one plus the number of divisors of N ...

Homotopies and Continuation

predictor-corrector methods

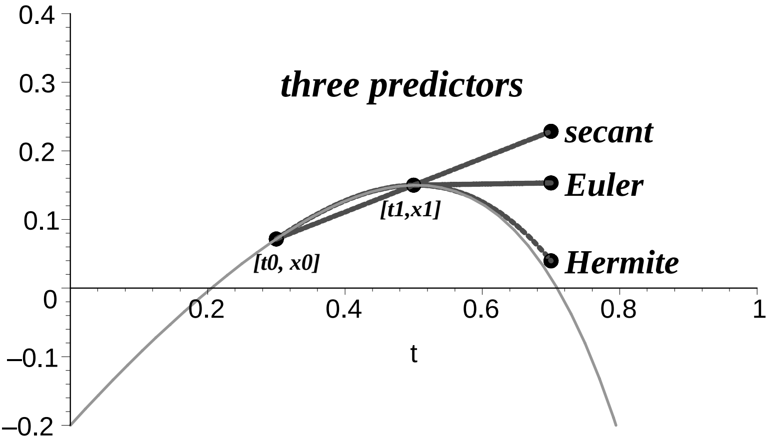
Consider $h(\mathbf{x}(t), t) = 0$. How does $\mathbf{x}$ change as t change? Apply $\frac{\partial}{\partial t}$ on h, with chain rule:

$$\frac{\partial h_k}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial t} + \frac{\partial h_k}{\partial t} = 0, \quad \frac{\partial h_k}{\partial \mathbf{x}} = \left[\frac{\partial h_k}{\partial x_1} \frac{\partial h_k}{\partial x_2} \cdots \frac{\partial h_k}{\partial x_n} \right], k = 1, 2, \ldots, n.$$

Denote $\Delta \mathbf{x} := \frac{\partial \mathbf{x}}{\partial t}$. After incrementing $t := t + \Delta t$, fix t and solve the linear system $\frac{\partial h}{\partial \mathbf{x}} \Delta \mathbf{x} = -\frac{\partial h}{\partial t}$ to obtain $\Delta \mathbf{x}$, the tangent to the path.

For some step size $\lambda > 0$, the updates $\mathbf{x} := \mathbf{x} + \lambda \Delta \mathbf{x}$ give the Euler predictor.

three types of predictors



increment-and-fix continuation

Since t is an artificial parameter, after prediction the value for t stays fixed during correction.

An increment-and-fix continuation method consists of

1. extrapolation to predict the solution,
2. Newton's method is used as corrector, and
3. an adaptive step-length control strategy.

Cost of the method depends on

1. cost to evaluate polynomials and derivatives,
2. cost to solve a linear system, and
3. number of steps to take along a path.

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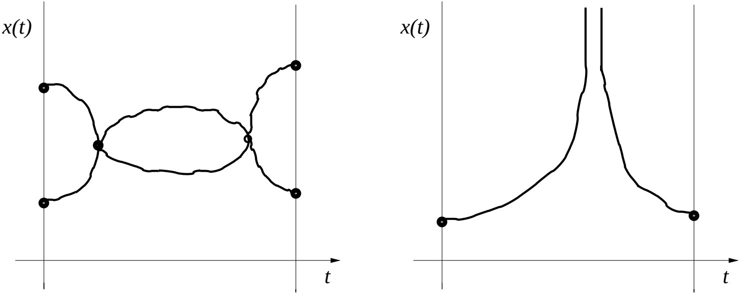
what can go wrong

Below are schematics of bad solution paths:



We worry about

1. singular points along a solution path;
2. paths diverging to infinity for $t < 1$.

Homotopies and Continuation

a random complex constant

The system $f(\mathbf{x}) = \mathbf{0}$ has total degree $D = \prod_{i=1}^{n} \deg(f_i)$.

Let $g(\mathbf{x}) = \mathbf{0}$ be a start system, $\deg(g_i) = \deg(f_i)$ for all i.

A linear artificial parameter homotopy is

$$h(\mathbf{x}, t) = \gamma(1-t)g(\mathbf{x}) + t\, f(\mathbf{x}) = 0, \quad \gamma \in \mathbb{C}, \quad t \in [0,1].$$

The complex constant γ is chosen at random.

Except for bad choices for γ belonging to a set of measure zero:

1. all solutions to $h(\mathbf{x}, t) = \mathbf{0}$ are regular; and
2. the #solutions in $\mathbb{C}^n$ equals exactly D,

for all t: $0 \leq t < 1$.

Homotopies and Continuation

no singularities along the paths

Theorem (regularity of the solution paths)

Let $f(\mathbf{x}) = \mathbf{0}$ have total degree D,
$g(\mathbf{x}) = \mathbf{0}$ be a start system based on D, and
$h(\mathbf{x}, t) = \gamma(1-t)g(\mathbf{x}) + t\, f(\mathbf{x}) = \mathbf{0}$, $\gamma \in \mathbb{C}$, $t \in [0,1]$.
Except for a set of measure zero, a set of bad choices for γ, the Jacobian matrix of $h(\mathbf{x}, t) = \mathbf{0}$ has full rank for t: $0 \leq t < 1$.

Proof. Let J_h be the Jacobian matrix of $h(\mathbf{x}, t) = \mathbf{0}$ and

$$E(\mathbf{x}, t) = \left\{ \begin{array}{r} h(\mathbf{x}, t) = \mathbf{0} \\ \det(J_h(\mathbf{x}, t)) = 0. \end{array} \right.$$

At $t = t^*$: $E(\mathbf{x}^*, t^*) = \mathbf{0}$, $\mathbf{x}^*$ is singular.

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apply elimination theory

To apply the main theorem of elimination theory:

- Embed $E(\mathbf{x}, t)$ in projective space $\mathbb{P}^{n+1}$.
- Eliminate all variables $\mathbf{x}$ from $E(\mathbf{x}, t) = \mathbf{0}$.

Result: a polynomial p in one variable t vanishing at those t for which the $\mathbf{x}$ is a singular solution of $h(\mathbf{x}, t) = \mathbf{0}$.

- $p(t) \not\equiv 0$, $h(\mathbf{x}, 0) = g(\mathbf{x}) = \mathbf{0}$ has only regular solutions.
- Complex γ gives complex solutions of $p(t) = 0$.

Except for a set of bad choices for γ, a set of measure zero, all roots of $p(t)$ will miss $[0, 1]$. □

To show that all paths are bounded, embed $h(\mathbf{x}, t) = \mathbf{0}$ in $\mathbb{P}^{n+1}$ and add $x_0 = 0$ to h to form $E(\mathbf{x}, t) = \mathbf{0}$.

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Summary + Exercises

We defined homotopies to solve systems based on their total degree and explained continuation methods. With a random constant we obtain regularity and boundedness.

Exercises:

1. Write an algorithm to enumerate all solutions of start system based on the total degree. Modify it so you can compute only the k-th solution, for k any number between 1 and D.
2. For the homotopy on the first slide to solve two quadrics, use elimination methods to verify that there are singular solutions for $t \approx 0.92$.

more exercises

3. Consider the homotopy

$$h(x,y,t) = \left(\left\{ \begin{array}{c} x^2 - 1 \\ y^2 - 1 \end{array} \right. \right) (1-t) + \left(\left\{ \begin{array}{c} y^2 - 1 \\ x^2 - 3 \end{array} \right. \right) t = \mathbf{0}.$$

   For which values of t do we have diverging paths? Show that with a random complex constant γ in $h(x,y,t) = \mathbf{0}$ there are no divergent paths with probability one.

4. Write a Maple procedure or code in Sage to generate the equations for the problem in magnetics.

5. Download phc from `http://www.math.uic.edu/~jan/download.html`. Use it to solve the equations for the problem in magnetics for $N = 10$. How many real solutions does the system have?

and more exercises

6. Use phc to track the paths defined by the homotopy on the first slide (with $\gamma = 1$). Describe what happens. Solve the system with random γ. The system solved by this homotopy has two distinct roots, one of the roots occurs with multiplicity higher than one. Can you tell from the output of phc which root is multiple?
7. The final observation made in this lecture concerned the number of paths converging to a multiple root. Try to formalize this statement, using a perturbation argument to prove how to count roots with their multiplicities.

Homework collected on Friday 28 January in class at 10AM: exercises 1, 5, 7 of Lecture 1; exercises 4, 5, 6, 7 of Lecture 2; and exercises 2, 3, 5 of Lecture 3.