

## Real Homotopies

In many applications, the polynomial systems have real coefficients and only the real solutions matter. To solve such systems we can use real homotopies if we then know how to deal with quadratic turning points [3]. The application we consider is the  $n$ -body problem from celestial mechanics [1].

### 1 Quadratic turning points

All polynomials in this lecture have real coefficients. Therefore, the roots appear in conjugated pairs, as stated in the following proposition.

**Proposition 1.1**  $p \in \mathbb{R}[x]: p(z) = 0 \Rightarrow p(\bar{z}) = 0$ .

*Proof.* Denote  $p$  as  $p(x) = \sum_{i=0}^d c_i x^i$ ,  $c_i \in \mathbb{R}$ . Then

$$\overline{p(x)} = \sum_{i=0}^d \overline{c_i x^i} = \sum_{i=0}^d \overline{c_i} \overline{x^i} = \sum_{i=0}^d c_i \overline{x^i} = p(\overline{x}).$$

If  $p(z) = 0$ , then  $\overline{p(z)} = 0$  and therefore  $p(\bar{z}) = 0$ . □

A real homotopy  $h(\mathbf{x}, t) = \mathbf{0}$  is defined by the map

$$h: \mathbb{C}^n \times \mathbb{R}^1 \rightarrow \mathbb{C}^n: (\mathbf{x}, t) \mapsto h(\mathbf{x}, t), \quad h \in \mathbb{R}[\mathbf{x}, t]. \quad (1)$$

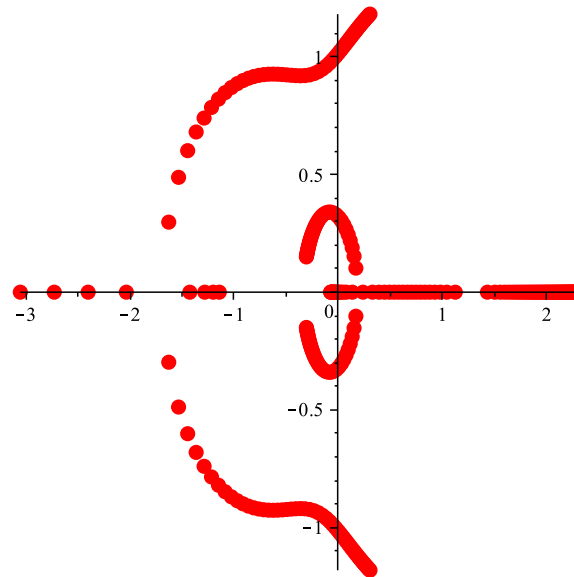
We consider square systems of  $n$  polynomials with real coefficients in  $n$  unknowns  $\mathbf{x}$ . Although we are mainly interested in polynomial systems, the path following methods apply to analytic systems as well.

The homotopies we considered so far always contained random complex constants. Thanks to this random complex constant  $\gamma$ , we could avoid singular solutions along the paths. If we no longer use random complex constants, singularities become unavoidable. The simplest types of singular solutions are quadratic turning points.

Below is Maple code to illustrate the behaviour of solution trajectories defined by real homotopies. We generate two random quintics – one root is always real – and look at the solutions of a homotopy involving these two polynomials. Fixing the seed should always render the same polynomials (done with Maple 14 on Mac OS X), but just in case we also list the two random polynomials below.

```
[> r := () -> op(RandomTools[Generate]([float(range = -1..1)])):
[> randomize(10): # fix the seed
[> p := randpoly(x, coeffs=r);
      5          4          3          2
-0.068604539 - 0.165313289 x - 0.283169405 x + 0.910871068 x + 0.425985216 x - 0.902734705 x
[> q := randpoly(x, coeffs=r);
      5          4          3          2
 0.101458438 - 0.425548444 x + 0.642135843 x + 0.034505744 x + 0.891888227 x + 0.149326479 x
[> h := t -> (1-t)*q + t*p:
[> L := [seq([fsolve(h(k/30), x, complex)], k=-30..30)]:
[> A := seq(plots[complexplot](L[k], style=point, symbol=solidcircle, symbolsize=20), k=1..nops(L)):
[> plots[display](A);
[> plots[display](A, insequence=true);
```

The last command produces an animation, while the result of the next to last plot command is shown next. The conjugation of the roots is apparent by the symmetry of the plot.



In the definition below (adapted from [3], see also [2]), by  $\partial_{\mathbf{x}}$  and  $\partial_t$  we denote the partial derivatives with respect to  $\mathbf{x}$  and  $t$ .

**Definition 1.1** The point  $(\mathbf{x}^*, t^*) \in \mathbb{C}^n \times \mathbb{R}^1$  is a *quadratic turning point* of  $h(\mathbf{x}, t) = \mathbf{0}$  if

1.  $h(\mathbf{x}^*, t^*) = \mathbf{0}$ ;
2.  $\text{rank}(\partial_{\mathbf{x}}h(\mathbf{x}^*, t^*)) = n - 1$ ;
3.  $[\partial_{\mathbf{x}}h(\mathbf{x}^*, t^*) \quad \partial_t h(\mathbf{x}^*, t^*)]$  has real rank  $2n - 1$ ; and
4. for all  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n \setminus \{0\}$  satisfying  $\mathbf{u}^T \partial_{\mathbf{x}}h(\mathbf{x}^*, t^*) = \mathbf{0}$  and  $\partial_{\mathbf{x}}h(\mathbf{x}^*, t^*)\mathbf{v} = \mathbf{0}$ , we have

$$\mathbf{u}^T \partial_{\mathbf{xx}}^2 h(\mathbf{x}^*, t^*) \mathbf{v} \mathbf{v} = 0, \quad (2)$$

where  $\partial_{\mathbf{xx}}^2 h$  contains all second order derivatives with respect to  $\mathbf{x}$  of  $h$ .

We characterize quadratic turning points geometrically and algebraically, following two propositions in [3]. At a quadratic turning points, real paths turn complex and complex paths turn real, as expressed in the following

**Proposition 1.2 (Proposition 2.1 in [3])**  $(\mathbf{x}^*, t^*)$  is a quadratic turning point if

1. there are exactly two paths  $\Gamma_1$  and  $\Gamma_2$  passing through  $(x^*, t^*)$ ,  $\Gamma_1$  is real,  $\Gamma_2$  is complex;
2. the tangent vectors at  $(x^*, t^*)$  to  $\Gamma_1$  and  $\Gamma_2$  are perpendicular to each other:  
if  $\phi$  is tangent to  $\Gamma_1$ , then  $i\phi$  is tangent to  $\Gamma_2$ ;
3.  $\Gamma_1$  and  $\Gamma_2$  lie on opposite sides of  $(x^*, t^*)$  with respect to the  $t$ -direction:  
the second derivatives of the  $t$ -component of  $\Gamma_1$  and  $\Gamma_2$  have different sign at  $(\mathbf{x}^*, t^*)$ .

For the algebraic definition of a quadratic turning point we use another proposition of [3], formulated as

**Proposition 1.3 (Proposition 3.1 in [3])**  $(\mathbf{x}^*, t^*)$  is a quadratic turning point of the homotopy  $h(\mathbf{x}, t)$  if and only if  $(\mathbf{x}^*, t^*)$  is a regular solution of

$$H(\mathbf{x}, t) = \begin{cases} h(\mathbf{x}, t) = \mathbf{0} \\ \det(\partial_{\mathbf{x}}h(\mathbf{x}, t)) = 0 \end{cases} \quad (3)$$

where  $\partial_{\mathbf{x}}h(\mathbf{x}, t)$  is the matrix of all partial derivatives of  $h$  with respect to the variables in  $\mathbf{x}$ .

Our algebraic definition justifies the designation of quadratic turning points as the simplest type of singular solutions. Already by adding one condition to the homotopy, the augmented system has the quadratic turning point as a regular solution.

To deal with singularities along solution paths we distinguish two stages, called detection and location. In the detection stage we look for a criterion for the singularity. By location we mean the accurate location of the singular solution.

## 2 The Newtonian $n$ -body problem

Consider  $n$  point particles with masses  $m_i$  and positions at  $\mathbf{x}_i \in \mathbb{R}^d$ ,  $i = 1, 2, \dots, n$ . The positions of the particles obey Newton laws:

$$m_j \frac{\partial^2 \mathbf{x}_j(t)}{\partial t^2} = \sum_{i \neq j} \frac{m_i m_j (\mathbf{x}_i(t) - \mathbf{x}_j(t))}{r_{ij}^3}, \quad j = 1, 2, \dots, n, \quad (4)$$

where  $r_{ij}$  is the distance between  $\mathbf{x}_i(t)$  and  $\mathbf{x}_j(t)$  at time  $t$ .

In  $\mathbb{R}^2$ , a relative equilibrium motion is a solution of (4) of the form  $\mathbf{x}_i(t) = R(t)\mathbf{x}_i(0)$  where  $R(t)$  is a rotation with angular velocity  $v > 0$  around the center  $\mathbf{c} \in \mathbb{R}^2$ . Such a solution must satisfy

$$\lambda(\mathbf{x}_j - \mathbf{c}) = \sum_{i \neq j} \frac{m_i (\mathbf{x}_i - \mathbf{x}_j)}{r_{ij}^3}, \quad j = 1, 2, \dots, n, \quad (5)$$

where  $\lambda = -v^2 < 0$ .

Multiplying the  $j$ th equation in (5) by  $m_j$  and summing gives  $m\mathbf{c} = \sum_{j=1}^n m_j \mathbf{x}_j$ , where  $m$  is the sum of the  $n$  masses  $m_j$ ,  $j = 1, 2, \dots, n$ . For  $m > 0$ , we set  $\lambda' = \lambda/m$  and the central configuration equations become

$$\sum_{i=1}^n m_i S_{ij}(\mathbf{x}_i - \mathbf{x}_j) = 0, \quad j = 1, 2, \dots, n, \quad (6)$$

where

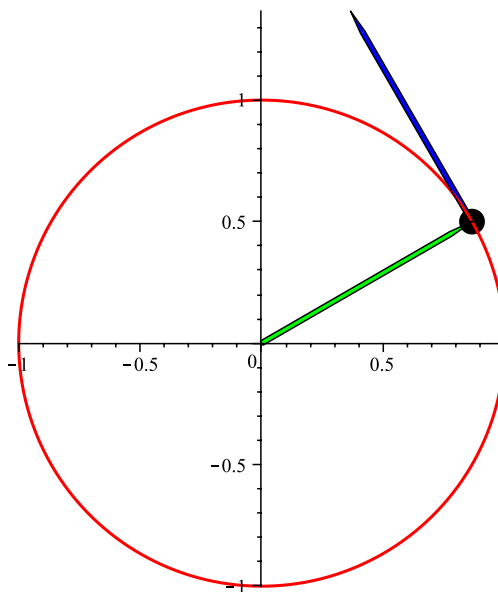
$$S_{ij} = \frac{1}{r_{ij}^3} + \lambda', \quad \text{for } i \neq j, \quad \text{and } S_{ii} = 0. \quad (7)$$

For  $n = 3$  and  $d = 2$ , we consider three points in the plane and have six unknowns for their coordinates. In this case, formula (6) generates a system of six equations in the unknown coordinates, with parameters  $m_1$ ,  $m_2$  and  $m_3$ . Following [1], the center of mass  $\mathbf{c}$  can be chosen so that  $\lambda' = -1$ .

## 3 Arc length parameter continuation

Consider the unit circle defined by the single polynomial equation  $f(x_1, x_2) = x_1^2 + x_2^2 - 1 = 0$ . A natural way to parametrize the solution set is to use  $(x_1 = \cos(\theta), x_2 = \sin(\theta))$ . Letting  $\theta$  range between 0 and  $2\pi$  then traces all points on the circle. The tangent at the point  $(\cos(\theta), \sin(\theta))$  is  $\pm(\sin(\theta), -\cos(\theta))$ . Walking the circle in the  $+$  direction of the tangent will trace the circle counterclockwise, while the  $-$  direction goes

clockwise. Subdividing the interval  $[0, 2\pi]$  using step size  $\Delta\theta$  as multiplier for the tangent vector leads to a tiling of the arc with line segments. As  $\Delta\theta$  gets smaller, the sum of the lengths of those line segment will converge to the length of the arc traced along the path. If we stay real, then every point on the circle is a quadratic turning point.



Generalizing the arc length parameter concept, we take the circle again, but now we sweep it by a line, using the homotopy

$$h(\mathbf{x}, t) = \begin{cases} x_1^2 + x_2^2 - 1 = 0 \\ (1-t)(x_1 + 2) + t(x_1 - 2) = 0 \end{cases} \quad t \in [0, 1]. \quad (8)$$

As  $t$  moves from 0 to 1,  $x_1$  goes from  $-2$  to  $+2$ , as the line  $x_1 = -2 + 4t$  sweeps the circle, we find  $x_2 = \pm\sqrt{-3 + 16t - 16t^2}$  as the two solutions.

The arc length parametrization of a homotopy  $h(\mathbf{x}, t) = \mathbf{0}$  introduces a new parameter  $s$  to describe the path  $(\mathbf{x}(s), t(s))$ . At any point along the path  $(\mathbf{x}_1, t_1)$ , we compute the tangent vector via application of the chain rule:

$$\frac{dh}{ds}(h(\mathbf{x}(s), t(s))) \equiv \mathbf{0} \quad \Rightarrow \quad \begin{bmatrix} \frac{\partial h}{\partial x} & \frac{\partial h}{\partial t} \end{bmatrix} \begin{bmatrix} \frac{d\mathbf{w}}{ds} \end{bmatrix} = \mathbf{0}, \quad \mathbf{w}(s) = (\mathbf{x}(s), t(s)). \quad (9)$$

In more compact form, at  $(\mathbf{x}_1, t_1)$ , we solve a homogeneous linear system

$$[h_{\mathbf{x}}(x_1, t_1) \quad h_t(x_1, t_1)] \begin{bmatrix} \frac{d\mathbf{w}}{ds} \end{bmatrix} = \mathbf{0}, \quad (10)$$

of  $n$  equations in  $n+1$  unknowns to find the value of the tangent vector  $\frac{d\mathbf{w}}{ds}$  at  $(\mathbf{x}_1, t_1)$ , denoted by  $\frac{d\mathbf{w}}{ds}(\mathbf{x}_1, t_1)$ . Up to scaling this linear system has a unique solution if and only if the point  $(\mathbf{x}_1, t_1)$  is a regular solution of  $h(\mathbf{x}, t) = \mathbf{0}$ .

To get a unique tangent vector at  $(\mathbf{x}_1, t_1)$  we first divide by its length so that so that  $|\frac{d\mathbf{w}}{ds}(\mathbf{x}_1, t_1)| = 1$ . Then we have  $\pm\frac{d\mathbf{w}}{ds}(\mathbf{x}_1, t_1)$  and must decide whether to choose  $+$  or  $-$  in the orientation. We compare the angle between  $\pm\frac{d\mathbf{w}}{ds}(\mathbf{x}_1, t_1)$  and the previous tangent vector  $\frac{d\mathbf{w}}{ds}(\mathbf{x}_0, t_0)$ . The correct orientation is to ensure that the angle between two consecutive tangent vectors is acute. Following [5], we apply the rule

$$\text{if } \left\langle \frac{d\mathbf{w}}{ds}(\mathbf{x}_0, t_0), \frac{d\mathbf{w}}{ds}(\mathbf{x}_1, t_1) \right\rangle < 0, \text{ then } \frac{d\mathbf{w}}{ds}(\mathbf{x}_1, t_1) := -\frac{d\mathbf{w}}{ds}(\mathbf{x}_1, t_1), \quad (11)$$

where  $\langle \cdot, \cdot \rangle$  denotes the usual inner product of two vectors. The computed tangent vector can then be used to predict the next solution  $(\widehat{\mathbf{x}}_2, \widehat{t}_2)$  along the path:

$$(\widehat{\mathbf{x}}_2, \widehat{t}_2) := (\mathbf{x}_1, t_1) + \lambda \frac{d\mathbf{w}}{ds}(\mathbf{x}_1, t_1), \quad (12)$$

where  $\lambda > 0$  is the step size.

As the artificial parameter  $t$  moves in its prescribed range, like  $[-2, +2]$  for the example (8), the arc length parameter  $s$  takes into account the actual geometry of the path. When approaching a turning point, the increments  $\Delta t(s)$  will decrease gradually. When  $\Delta t(s)$  changes sign, i.e.:  $\Delta t(s)$  is a decrement, then we have passed the turning point. The condition  $\Delta t(s) = 0$  can be used to locate the quadratic turning point precisely, or otherwise one may also use the augmented system in (3) or some variant of it to compute the quadratic turning point.

During the correction, the parameter  $t$  does not stay fixed. Recall that  $h(\mathbf{x}, t) = \mathbf{0}$  has  $n$  equations in  $n + 1$  unknowns:  $\mathbf{x}$  and  $t$ . Denote the tangent vector as  $T = [T_{\mathbf{x}} \ T_t]$ , with  $T_{\mathbf{x}}$  the first  $n$  components for  $\mathbf{x}$  and the last component  $T_t$  for  $t$ . The correction term  $(\Delta \mathbf{x}, \Delta t)$  is then computed via Newton's method to be perpendicular to the tangent:

$$\begin{bmatrix} \partial_{\mathbf{x}} h & \partial_t h \\ T_{\mathbf{x}} & T_t \end{bmatrix} \begin{bmatrix} \Delta \mathbf{x} \\ \Delta t \end{bmatrix} = \begin{bmatrix} -h \\ 0 \end{bmatrix} \quad (13)$$

Evaluating  $h$  and the partial derivatives in the iteration at  $(\mathbf{x}^{(k)}, t^{(k)})$ , the next update is computed as  $(\mathbf{x}^{(k+1)}, t^{(k+1)}) := (\mathbf{x}^{(k)}, t^{(k)}) + (\Delta \mathbf{x}, \Delta t)$ . Notice that the third condition in the definition of a quadratic turning point guarantees that the matrix of the linear system in (13) has full rank and the update vector is uniquely determined.

To pass through the turning point, switching from a real to a complex path (or vice versa), we can use the geometric definition in Proposition 1.2.

Figure 1 shows a schematic of a tangent predictor, followed by a perpendicular correction before and after a turning point. With a shooting method [4] we can determine the right step size along a tangent to arrive at the location of a quadratic turning point. The objective is to locate a point on the curve where the tangent is perpendicular to the  $t$ -axis.

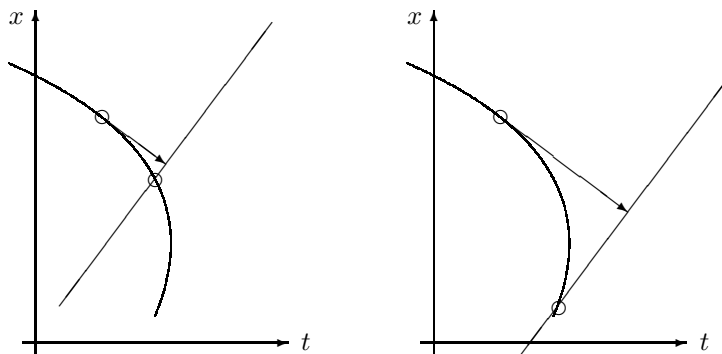


Figure 1: Shooting for a quadratic turning point. At the left, the point after the correction still has a tangent increasing in the horizontal direction while at the right the tangent at the new point is pointing backward.

## 4 A generic turn

The main result of [3] is described in the theorem below.

**Theorem 4.1** Let  $f(\mathbf{x}) = \mathbf{0}$  and  $g(\mathbf{x}) = \mathbf{0}$  respectively be a target and start system of  $n$  polynomial equations with real coefficients. Denote  $d_i = \deg(f_i) = \deg(g_i)$ ,  $i = 1, 2, \dots, n$ . Consider

$$h_i(\mathbf{a}, \mathbf{b}, \mathbf{x}, t) = (1 - t)g_i + tf_i + t(1 - t)r_i, \quad i = 1, 2, \dots, n, \quad (14)$$

with

$$r_i(\mathbf{a}, \mathbf{b}, \mathbf{x}) = \sum_{j=1}^n a_j^i x_j^{d_i} + \sum_{j \neq k} b_{jk}^i x_j x_k^{d_i-1}, \quad i = 1, 2, \dots, n. \quad (15)$$

For generic choices of the real coefficients in  $\mathbf{a}$  and  $\mathbf{b}$ ,  $h(\mathbf{x}, t) = \mathbf{0}$  has no singular solutions other than a finite number of quadratic turning points.

## 5 Exercises

1. Run the Maple animation on the complex root plot of the quintic polynomials. Even better: make a version of the animation with Sage. Compute the speed of the roots in function of  $t$  and make a plot of the evolution of this speed.
2. For the homotopy  $h(\mathbf{x}, t) = \mathbf{0}$  in (8), the two quadratic turning points are  $\pm(1, 0)$ . Verify the definition and the geometric and algebraic characterizations in Proposition 1.2 and Proposition 1.3.
3. Use Maple or Sage to generate the system defined by (6) and (7), for  $n = 3$  and  $d = 2$ . Consider the special case when all masses are equal. Describe the symmetric structure of this system. How many solutions do you expect? Solve it.
4. Justify the rule (11) to determine the orientation of the tangent vectors of two consecutive points along a path to ensure the angle between the tangent vectors is acute.
5. Consider the homotopy  $h(\mathbf{x}, t) = \mathbf{0}$  in (8) at the quadratic turning point  $(1, 0)$ . Apply Newton's method on the system  $H(\mathbf{x}, t) = \mathbf{0}$ , starting at points close to  $(1, 0)$ . Describe what happens.
6. Consider Figure 1 to apply shooting to locate the location of a quadratic turning point. Two predictions with different step sizes, say  $h_1$  and  $h_2$ , lead to a solution before and after the quadratic turning point. Use interpolation to derive a new step size  $h$ . Explain how this leads to a shooting method to locate the quadratic turning point.

## References

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- [2] M.E. Henderson and H.B. Keller. Complex bifurcation from real paths. *SIAM Journal on Applied Mathematics*, 50(2):460–482, 1990.
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- [5] A. Morgan. *Solving polynomial systems using continuation for engineering and scientific problems*. Prentice-Hall, 1987.