Multiplicity Structure

Following [2], we define the multiplicity via differential conditions and discuss a symbolic-numeric method to compute the multiplicity structure at an isolated singular solution, using differential operators. The dual view on ideals originated in the work of Macaulay, see [4], [6], and [5].

1 Differentials and Duality

A polynomial p in one variable x has an m-fold zero at z if

1. $p(x) = (x-z)^m q(x)$ and $q(z) \neq 0$; or
2. $p(z) = 0$ and $\frac{\partial^k p}{\partial x^k}(z) = 0$, for $k = 1, 2, \ldots, m-1$.

In both cases, we say that m is the multiplicity of the zero z. For polynomial systems, standard bases generalize the first definition of an m-fold root. In this lecture we will generalize the second definition using differential operators. See [4] for a discussion of symbolic methods to represent multiplicities.

For a zero $\mathbf{z} \in \mathbb{C}^n$ and a natural vector $\mathbf{a} \in \mathbb{N}^n$, the differential operator $\partial_\mathbf{a}[\mathbf{z}]$ is defined by

$$\partial_\mathbf{a}[\mathbf{z}] : \mathbb{C}[\mathbf{x}] \rightarrow \mathbb{C} : p \mapsto (\partial_\mathbf{a} p)(\mathbf{z}), \quad \text{with} \quad \partial_\mathbf{a} p = \frac{1}{a_1! a_2! \cdots a_n!} \frac{\partial^{a_1+a_2+\cdots+a_n} p}{\partial x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}}. \tag{1}$$

With this notation, we can write the Taylor expansion of p about $\mathbf{z}$ in a very compact way. More importantly, observe the linearity of the differential operator: $\partial_\mathbf{a}[\mathbf{z}](\lambda p + \mu q) = \lambda \partial_\mathbf{a}[\mathbf{z}](p) + \mu \partial_\mathbf{a}[\mathbf{z}](q)$.

A general differential operator is a linear combination of several $\partial_\mathbf{a}[\mathbf{z}]$'s. The dual space $D_\mathbf{z}[I]$ of I at $\mathbf{z}$ is

$$D_\mathbf{z}[I] = \left\{ d[\mathbf{z}] = \sum_\mathbf{a} c_\mathbf{a} \partial_\mathbf{a}[\mathbf{z}], c_\mathbf{a} \in \mathbb{C} \mid d[\mathbf{z}](p) = 0, \forall p \in I \right\}. \tag{2}$$

It is the space of all differential operators which make every polynomial in the ideal I vanish at the root $\mathbf{z}$.

Consider $I = \langle x_1^2, x_1 x_2, x_2^2 \rangle$, $\mathbf{z} = (0,0)$. Take $p \in I$ as $p = q_{20} x_1^2 + q_{11} x_1 x_2 + q_{02} x_2^2$, $q_{ij} \in \mathbb{C}[\mathbf{x}]$.

$$\partial_{10}(p) = x_1 \left(\frac{\partial q_{20}}{\partial x_1} x_1 + 2 q_{20} \right) + \left(\frac{\partial q_{11}}{\partial x_1} x_1 x_2 + x_2 q_{11} \right) + \frac{\partial q_{02}}{\partial x_1} x_2^2, \quad \partial_{10}[(x_1 = 0, x_2 = 0)](p) = 0. \tag{3}$$

By $x_1 \leftrightarrow x_2$: $\partial_{01}[\mathbf{z}](p) = 0$. Higher derivatives have nonzero constants and do not vanish at $\mathbf{z}$. Thus, $D_\mathbf{z}[I] = \text{span}\{ \partial_{00}[\mathbf{z}], \partial_{10}[\mathbf{z}], \partial_{01}[\mathbf{z}] \}$. Since $\dim(D_\mathbf{z}[I]) = 3$, the multiplicity of $\mathbf{z}$ equals three.

We illustrate the correspondence between standard bases and dual space via an example from [5]:

$$I = \langle x_1^2 + 2x_2^2 - 2x_2, x_1 x_2^2 - x_1 x_2, x_2^3 - 2x_2^2 + x_2 \rangle \quad \text{with} \quad \mathbf{z}_0 = (0,0) \quad \text{and} \quad \mathbf{z}_1 = (0,1) \tag{4}$$

two roots with respective multiplicities $m_0 = 2$ and $m_1 = 3$. The dual space $D[I]$ of I is $D_{\mathbf{z}_0}[I] \cup D_{\mathbf{z}_1}[I]$. The dual space of I at $\mathbf{z}_0$ is $D_{\mathbf{z}_0}[I] = \text{span}\{\partial_{00}[\mathbf{z}_0], \partial_{10}[\mathbf{z}_0]\}$ and $D_{\mathbf{z}_1}[I] = \text{span}\{\partial_{00}[\mathbf{z}_1], \partial_{10}[\mathbf{z}_1], 2\partial_{20}[\mathbf{z}_1] - \partial_{01}[\mathbf{z}_1]\}$. The elements of a basis for the quotient ring are illustrated in Figure 1.

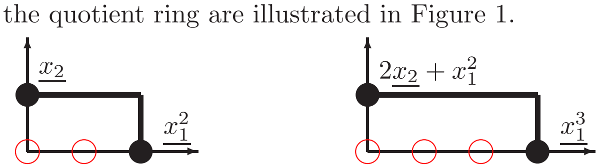

Figure 1: Standard bases for $\mathbf{z}_0$ and $\mathbf{z}_1$ (after shift) for the ideal in (4). For every leading term $\partial_\mathbf{a}$ in a generator of $D_\mathbf{z}[I]$, there is a corresponding standard monomial $\mathbf{x}^\mathbf{a}$, where $\mathbf{a}$ is marked by a dark circle.

2 The Cyclic 9-Roots Problem

The system for the cyclic 9-roots problem is

$$f(\mathbf{x}) = \left\{ \begin{array}{rcl} x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 & = & 0 \\ x_1x_2 + x_1x_9 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_6 + x_6x_7 + x_7x_8 + x_8x_9 & = & 0 \\ x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_6 + x_5x_6x_7 & & \\ + \, x_6x_7x_8 + x_7x_8x_9 + x_8x_9x_1 + x_9x_1x_2 & = & 0 \\ x_1x_2x_3x_4 + x_2x_3x_4x_5 + x_3x_4x_5x_6 + x_4x_5x_6x_7 + x_5x_6x_7x_8 & & \\ + \, x_6x_7x_8x_9 + x_7x_8x_9x_1 + x_8x_9x_1x_2 + x_9x_1x_2x_3 & = & 0 \\ x_1x_2x_3x_4x_5 + x_2x_3x_4x_5x_6 + x_3x_4x_5x_6x_7 + x_4x_5x_6x_7x_8 + x_5x_6x_7x_8x_9 & & \\ + \, x_6x_7x_8x_9x_1 + x_7x_8x_9x_1x_2 + x_8x_9x_1x_2x_3 + x_9x_1x_2x_3x_4 & = & 0 \\ x_1x_2x_3x_4x_5x_6 + x_2x_3x_4x_5x_6x_7 + x_3x_4x_5x_6x_7x_8 & & \\ + \, x_4x_5x_6x_7x_8x_9 + x_5x_6x_7x_8x_9x_1 + x_6x_7x_8x_9x_1x_2 & & \\ + \, x_7x_8x_9x_1x_2x_3 + x_8x_9x_1x_2x_3x_4 + x_9x_1x_2x_3x_4x_5 & = & 0 \\ x_1x_2x_3x_4x_5x_6x_7 + x_2x_3x_4x_5x_6x_7x_8 + x_3x_4x_5x_6x_7x_8x_9 & & \\ + \, x_4x_5x_6x_7x_8x_9x_1 + x_5x_6x_7x_8x_9x_1x_2 + x_6x_7x_8x_9x_1x_2x_3 & & \\ + \, x_7x_8x_9x_1x_2x_3x_4 + x_8x_9x_1x_2x_3x_4x_5 + x_9x_1x_2x_3x_4x_5x_6 & = & 0 \\ x_1x_2x_3x_4x_5x_6x_7x_8 + x_2x_3x_4x_5x_6x_7x_8x_9 + x_3x_4x_5x_6x_7x_8x_9x_1 & & \\ + \, x_4x_5x_6x_7x_8x_9x_1x_2 + x_5x_6x_7x_8x_9x_1x_2x_3 + x_6x_7x_8x_9x_1x_2x_3x_4 & & \\ + \, x_7x_8x_9x_1x_2x_3x_4x_5 + x_8x_9x_1x_2x_3x_4x_5x_6 + x_9x_1x_2x_3x_4x_5x_6x_7 & = & 0 \\ x_1x_2x_3x_4x_5x_6x_7x_8x_9 - 1 & = & 0. \end{array} \right. \tag{5}$$

Taking into account the permutation symmetry, the isolated solutions are generated by 333 regular roots and 162 roots of multiplicity four. Only one deflation is needed to locate every 4-fold root accurately.

Running the algorithm of Dayton and Zeng [2] (explained below) on one root (after deflation) yields

$$\begin{array}{l} H[1] = 1, H[2] = 2, H[3] = 1, H[4] = 0, \\ \text{with } H[i] = \text{Nullity}(S_i) - \text{Nullity}(S_{i-1}), i > 0, \end{array} \tag{6}$$

so we compute locally the multiplicity as 4.

3 Computing the Multiplicity Structure

In this section we summarize the key ideas of [2].

To compute the multiplicity of a solution $\mathbf{z}$, we compute the dimension of the dual space $D_{\mathbf{z}}[I]$. We look for differentiation functionals $d[\mathbf{z}] = \sum_{\mathbf{a}} c_{\mathbf{a}} \partial_{\mathbf{a}}[\mathbf{z}]$.

For $d[\mathbf{z}]$ to belong to $D_{\mathbf{z}}[I]$, $I = \langle f_i, i = 1, 2, \ldots, N \rangle$, we have the following membership criterion:

$$d[\mathbf{z}] \in D_{\mathbf{z}}[I] \Leftrightarrow d[\mathbf{z}](pf_i) = 0, \forall p \in \mathbb{C}[\mathbf{x}], i = 1, 2, \ldots, N. \tag{7}$$

To turn this criterion into an algorithm, observe:

1. since $d[\mathbf{z}]$ is linear, we may restrict p to $\mathbf{x}^{\mathbf{k}} = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$; and
2. we may limit the degrees $k_1 + k_2 + \cdots + k_n \leq a_1 + a_2 + \cdots + a_n$, as $\mathbf{z} = \mathbf{0}$ vanishes trivially if not annihilated by $\partial_{\mathbf{a}}$.

By these turns, we consider the application of a differential operator to monomial multiples of the polynomials generating the ideal, evaluated at the root $\mathbf{z}$.

Consider the sequence of *multiplicity matrices* S_k, where the columns of S_k are indexed by $\partial_{\mathbf{a}}$, for $|\mathbf{a}| \leq k$, and the rows by $\mathbf{x}^{\mathbf{b}} f_i$, for $|\mathbf{b}| \leq k-1$. The entries of S_k are the values of $\partial_{\mathbf{a}}(\mathbf{x}^{\mathbf{b}} f_i)$, evaluated at $\mathbf{z}$. By convention, $S_0 = f(\mathbf{z})$. The null space of these multiplicity matrices yield generators for the dual space $D_{\mathbf{z}}[I]$. Since the dual space is finitely generated, the algorithm stops when the nullity (dimension of the null space) does not increase between S_{k-1} and S_k. Formally we have

Theorem 3.1 (the Hilbert function) *The Hilbert function $H(k)$ and multiplicity m are*

$$H(k) = \text{nullity}(S_k) - \text{nullity}(S_{k-1}), k = 1, 2, \ldots \quad m = \sum_{k=1}^{\infty} H(k). \tag{8}$$

We take an example from [2]:

$$f(\mathbf{x}) = \begin{cases} f_1 = x_1 - x_2 + x_1^2 = 0 \\ f_2 = x_1 - x_2 + x_2^2 = 0 \end{cases} \quad \mathbf{z} = (0, 0). \tag{9}$$

The goal is the compute the multiplicity of $\mathbf{z}$. Below we show the consecutive multiplicity matrices.

		$\|a\|=0$	$\|a\|=1$		$\|a\|=2$			$\|a\|=3$			
		∂_{00}	∂_{10}	∂_{01}	∂_{20}	∂_{11}	∂_{02}	∂_{30}	∂_{21}	∂_{12}	∂_{03}
	f_1	0	1	–1	1	0	0	0	0	0	0
S_1	f_2	0	1	–1	0	0	1	0	0	0	0
	$x_1 f_1$	0	0	0	1	–1	0	1	0	0	0
	$x_1 f_2$	0	0	0	1	–1	0	0	0	1	0
	$x_2 f_1$	0	0	0	0	1	–1	0	1	0	0
S_2	$x_2 f_2$	0	0	0	0	1	–1	0	0	0	1
	$x_1^2 f_1$	0	0	0	0	0	0	1	–1	0	0
	$x_1^2 f_2$	0	0	0	0	0	0	1	–1	0	0
	$x_1 x_2 f_1$	0	0	0	0	0	0	0	1	–1	0
	$x_1 x_2 f_2$	0	0	0	0	0	0	0	1	–1	0
	$x_2^2 f_1$	0	0	0	0	0	0	0	0	1	–1
S_3	$x_2^2 f_2$	0	0	0	0	0	0	0	0	1	–1

As Nullity(S_2) = Nullity(S_3), the algorithm stops. That the multiplicity equals three follows from the number of generators of $D_{\mathbf{z}}[I] = \text{span}\{ \partial_{00}, \partial_{10} + \partial_{01}, -\partial_{10} + \partial_{20} + \partial_{11} + \partial_{02} \}$.

As in the case of polynomials in one variable and the deflation algorithm, the nullity (or the corank) of a matrix is computed via rank-revealing algorithms [3], where the QR decomposition is updated when new rows and/or columns are added to the matrix. Duality and structured linear algebra is described in [8], see also [7].

4 An Analysis of the Deflation Bound

The number of stages in the deflation method to recondition an isolated singular solution is bounded by the multiplicity. In [2], the authors introduce the *breadth* and *depth* of $D_{\mathbf{z}}[I]$ as

$$\text{breadth}_{\mathbf{z}}[I] = H(1) \quad \text{and} \quad \text{depth}_{\mathbf{z}}[I] = \max\{ \alpha \mid H(\alpha) > 0 \}, \tag{10}$$

where $H(\cdot)$ is the Hilbert function.

Theorem 4.1 (Dayton and Zeng [2]) *The number of deflation stages for an isolated solution $\mathbf{z} \in V(I)$ is bounded by* $\text{depth}_{\mathbf{z}}(I)$. *Furthermore, if k stages are executed in the deflation method, then 2^k differential functionals in $D_{\mathbf{z}}[I]$ have been computed by the deflation method.*

In [1], deflation has been extended to more general nonlinear systems

5 A Commutative Diagram

Assuming $\dim(V(I)) = 0$, Figure 2 shows a commutative diagram which appears in [9, Section 8.1]:

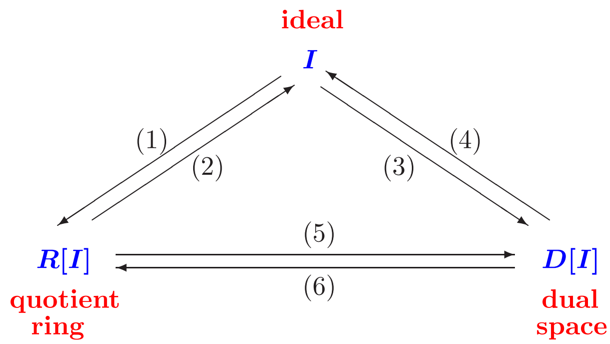

Figure 2: A commutative diagram between ideal I, the residue class ring $R[I]$ and the dual space $D[I]$.

(1) $\boldsymbol{I \rightarrow R[I]}$**:** the residue classes of Lagrange polynomials interpolating at $V(I)$ give a basis for $R[I]$.
$R[I] = \{ \ [p]_I \mid p \in \mathbb{C}[\mathbf{x}] \ \}$, where the residue class of p is $[p]_I = \{ \ r \in \mathbb{C}[\mathbf{x}] \mid p - r \in I \ \}$.

(2) $\boldsymbol{R[I] \rightarrow I}$**:** for some basis $\mathbf{b}$ of $R[I]$: $A_k \mathbf{b} = x_k \mathbf{b}$, $k = 1, 2, \ldots, n$, means $x_k \mathbf{b} = A_k \mathbf{b} \bmod I$, or $x_k \mathbf{b} - A_k \mathbf{b} = 0$ over $V(I)$, and $x_k \mathbf{b} - A_k \mathbf{b} \in I$ leads to a border basis for I.

(3) $\boldsymbol{I \rightarrow D[I]}$**:** evaluating at the zeroes $\mathbf{z} \in V(I)$ gives a natural basis for $D[I]$, m-fold zeroes are represented by m differential operators.

(4) $\boldsymbol{D[I] \rightarrow I}$**:** $I[D[I]] = \ker(D[I]) = \{ \ p \in \mathbb{C}[\mathbf{x}] : l(p) = 0, \forall l \in D[I] \ \}$

(5) $\boldsymbol{R[I] \rightarrow D[I]}$**:** $(R[I])^* = D[I]$. The monomials in a normal set (basis for the residue class ring $R[I]$) correspond to leading terms of differential operators, see Figure 1.

(6) $\boldsymbol{D[I] \rightarrow R[I]}$**:** $(D[I])^* = R[I]$. The leading terms of the differential operators that generate the dual space $D[I]$ correspond to basis elements for the quotient ring $R[I]$, again see Figure 1 for an example.

The utility of the dual space $D[I]$ for computing solutions seems to be limited as it comes as a consequence of solving the system $f(\mathbf{x}) = \mathbf{0}$. However, differential operators form a very convenient tool to describe multiple zeroes, as we have seen earlier. Recall that the computation of the multiplicity structure is a local calculation of the dual space at the zero.

6 Exercises

1. For a polynomial p in one variable with an m-fold zero at z prove

$$p(x) = (x-z)^m q(x) \text{ and } q(z) \neq 0 \quad \Leftrightarrow \quad p(z) = 0 \text{ and } \frac{\partial^k p}{\partial x^k}(z) = 0, \text{ for } k = 1, 2, \ldots, m-1, \qquad (11)$$

   i.e.: these are equivalent definitions of multiplicity m.

2. Take $I = \langle x_1^3 + x_1 x_2^2, x_1 x_2^2 + x_2^3, x_1^2 x_2 + x_1 x_2^2 \rangle$. Describe the dual space $D_{\mathbf{0}}[I]$.

3. Compare the multiplicity structure of the ideals $I_1 = \langle x_1^3, x_2^4 \rangle$ and $I_2 = \langle x_1^3, x_2^4 + x_1^2 x_2 \rangle$.
   (a) What is the multiplicity of $(0,0)$ for I_1 and I_2?
   (b) Compute $D_{\mathbf{0}}[I_1]$ and $D_{\mathbf{0}}[I_2]$.

4. For the computations of the previous exercise, make a cost analysis of the rank computations, taking into account the steady increase of dimensions of the matrices. Estimate the cost benefit of working incrementally, updating QR decompositions with new rows and columns, instead of recomputing each time from scratch.

5. Consider the system $f(x,y) = \left\{ \begin{array}{r} x^2 + y - 3 = 0 \\ x + 0.125y^2 - 1.5 = 0. \end{array} \right.$
   This system has one regular solution and a triple root at $(1,2)$.
   (a) Set up the multiplicity matrix at $(1,2)$ and use it to compute the multiplicity of this root.
   (b) Instead of $(1,2)$, evaluate the multiplicity matrix at $(0.9999999, 2.000000001)$. How large should the tolerance ϵ (used to decide the numerical rank) be in order to obtain correct results?

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