

## Hilbert Polynomials

For a monomial ideal, we derive the dimension counting the monomials in the complement, arriving at the notion of the Hilbert polynomial. The first half of the note is derived from [1, Chapter 9]. Following [2], cone decompositions of a ideal bound the complexity of Gröbner bases.

### 1 Dimension of the Variety of a Monomial Ideal

The variety of a monomial ideal  $I$  is a union of coordinate planes. The dimension of this variety is the highest dimension over all planes in this union. For  $n$  variables, this dimension equals  $n - m$ , where  $m$  is the minimal number of variables occurring in a generator  $\mathbf{x}^{\mathbf{a}}$  of  $I$ .

If  $I$  is zero dimensional, then the number of monomials not in  $I$  (visualized as the number of dots under the staircase representation of  $I$ ) equals the multiplicity of  $\mathbf{0} \in V(I)$ . For positive dimension  $I$ , we count the monomials not in  $I$ . For  $I = \langle x_1^{a_1} x_2^{a_2} \rangle$ , the complement of  $I$  consists of  $(a_1 + a_2)$  lines of monomials:  $\{ x_1^i x_2^j \mid i = 0, 1, \dots, a_1 - 1, j = 0, 1, \dots, a_2 - 1 \}$ . For a monomial ideal  $I$  generated by more than one monomial, see for example Figure 1, we take as  $b_1$  and  $b_2$  the respective smallest powers of  $x_1$  and  $x_2$  of the generators of  $I$  and count  $(b_1 + b_2)$  lines of monomials in addition to the finite number of monomials under the staircase, counting upwards starting at  $(b_1, b_2)$ .

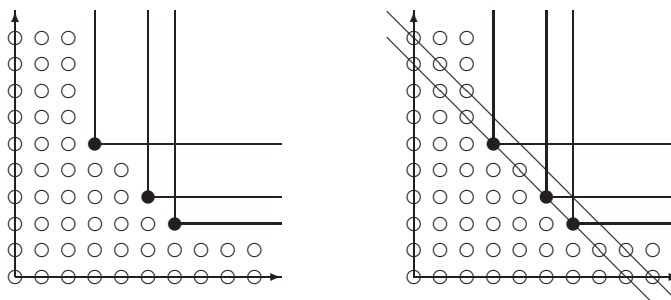


Figure 1: Black dots correspond to the generators of the monomial ideal  $I = \langle x_1^3 x_2^5, x_1^5 x_2^3, x_1^6 x_2^2 \rangle$ . Empty circles belong to the complement of  $I$ . Observe that dividing all generators by  $x_1^3 x_2^2$  will give an ideal that has  $\mathbf{0}$  as a 7-fold solution. For degree  $d \geq 9$ , the number of monomials in the complement of  $I$  of degree  $\leq d$  equals  $5d + 2 = (d + 1) + d + (d - 1) + (d + 1) + d + 7 - 6$  as shown by counting the monomials on the diagonal lines on the right staircase.

The calculation explained in the caption of Figure 1 generalizes for any monomial ideal in two variables, leading to a linear polynomial in the degree  $d$ . This polynomial is the Hilbert polynomial.

**Theorem 1.1** Consider a monomial ideal  $I$  in a polynomial ring of  $n$  variables with  $\dim V(I) = d$ . The number of monomials not in  $I$  of degree  $\leq s$ , for  $s$  sufficiently large, is a polynomial of degree  $d$  in  $s$  with positive coefficient of  $s^d$ .

To gain some intuition for why Theorem 1.1 might hold, consider ideals generated by one monomial. For example in 3 variables, consider  $I = \langle x_3^{a_3} \rangle$  and count all monomials not in  $I$  of degree  $\leq s$ . These monomials lie in  $a_3$  planes perpendicular to the third coordinate axis. Every plane holds  $s(s + 1)/2$  monomials  $\leq s$ , so the polynomial is  $a_3 s(s + 1)/2$ . For  $I = \langle x_1^{a_1} x_2^{a_2} x_3^{a_3} \rangle$ , we have  $(a_1 + a_2 + a_3)$  planes of monomials not in  $I$ .

For monomial ideals in 3 variables generated by more than one monomial, imagine a 3-dimensional generalization of Figure 1. As in 2 variables, we define powers  $b_i$  as the smallest powers of  $x_i$  in the generators of the monomial ideal. Then we count the monomials in the complement of  $\langle x_1^{b_1} x_2^{b_2} x_3^{b_3} \rangle$ . For sufficiently large degrees, we add the finite number of monomials under the staircase, starting the count at  $(b_1, b_2, b_3)$ .

**Proposition 1.1** For a monomial ideal  $I$  in a polynomial ring of  $n$  variables of dimension  $d$ , denote by  $h(s)$  the number of monomial ideals in  $I$  of degree  $\leq s$ , for  $s$  sufficiently large, then

$$h(s) = \sum_{i=0}^d a_{d-i} \binom{s}{d-i}, \quad a_i \in \mathbb{Z}, a_d > 0. \quad (1)$$

*Sketch of Proof.* That  $\deg(h(s)) = d$  follows from Theorem 1.1. Furthermore  $h(s)$  returns natural numbers when evaluated at natural values for  $s$ . Counting the number of monomials for  $d+1$  consecutive sufficiently large values of  $s$  leads to a special interpolation problem, solved by Gregory-Newton interpolation. The coefficients of the Gregory-Newton interpolating polynomial are binomial coefficients.  $\square$

## 2 The Dimension of a Variety

Given a term order  $>$ , a Gröbner basis  $G$  for an ideal  $I$  in  $\mathbb{K}[\mathbf{x}]$  gives a basis for the initial ideal  $\text{in}_>(I)$  as  $\{ \text{LT}_>(g) \mid g \in G \}$ . With  $G$  we have sufficiently many leading monomials for the division (or normal form) algorithm to give a unique outcome. If the Hilbert polynomial for  $\text{in}_>(I)$  is the same as the Hilbert polynomial for  $I$ , then with a Gröbner basis we have an algorithm to compute  $\dim(V(I))$ .

To reduce the dimension computation to counting monomials, we restrict our consideration to polynomials of bounded degree. Denote by  $\mathbb{K}[\mathbf{x}]_{\leq s} = \{ f \in \mathbb{K}[\mathbf{x}] \mid \deg(f) \leq s \}$  and

$$I_{\leq s} = \{ f \in I \mid \deg(f) \leq s \} \quad \text{and} \quad \text{in}_>(I)_{\leq s} = \{ f \in \text{in}_>(I) \mid \deg(f) \leq s \}. \quad (2)$$

With this notation, we can formally describe the “If” sentence as

$$h_I(s) = \dim \mathbb{K}[\mathbf{x}]_{\leq s} / I_{\leq s} \quad (3)$$

$$= \dim \mathbb{K}[\mathbf{x}]_{\leq s} / \text{in}_>(I)_{\leq s} \quad (4)$$

$$= h_{\text{in}_>(I)} \quad (5)$$

The equation between (3) and (4) means that the quotient rings have the same number of basis elements. Justifying this equation can happen via a Gröbner basis  $G_>$  for  $I$ . An argument by contradiction assumes a different number of basis elements for the quotient rings, which then conflicts with the property of  $G_>$  a Gröbner basis for  $I$ .

## 3 Using Macaulay 2

```
i1 : R = QQ[x,y];
i2 : I = ideal(x^3*y^5,x^5*y^3,x^6*y^2);
i3 : h = hilbertPolynomial I
o3 = 5*P
      0
o3 : ProjectiveHilbertPolynomial
i4 : H = hilbertPolynomial(I, Projective=>false)
o4 = 5
o4 : QQ[i]
```

## 4 The Complexity of Gröbner Bases

In [2], the following theorem is proven:

**Theorem 4.1** *Let  $I = \langle f_1, f_2, \dots, f_N \rangle$  be an ideal in  $\mathbb{K}[\mathbf{x}]$  and let  $d = \max_{i=1}^N \deg(f_i)$ . For any term order  $>$ , the degree of polynomials required in a Gröbner basis for  $I$  with respect to  $>$  is bounded by*

$$2 \left( \frac{d^2}{2} + d \right)^{2^{n-2}}. \tag{6}$$

As the Hilbert function counts the number of monomials in the complement of the ideal, it provides a measure for the monomials in a Gröbner basis of the ideal.

## 5 Cone Decompositions

Let  $\mathbf{u}$  be a subset of  $X = \{x_1, x_2, \dots, x_n\}$  and  $f \in \mathbb{K}[\mathbf{x}]$  a homogeneous polynomial. The set

$$C(f, \mathbf{u}) = \{ fg \mid g \in \mathbb{K}[\mathbf{u}] \} \tag{7}$$

is a *cone* spanned by  $f$  in  $\mathbf{u}$ .

The name for  $C(f, \mathbf{u})$  is justified by a graphical representation for two variables. For example  $C(x^2y^3, \emptyset)$  is just one dot at  $(2, 3)$ . The cone  $C(x^2y^3, \{x\})$  are all multiples for  $x^2y^3$ , depicted as a ray extending in the left  $x$ -direction. The cone  $C(x^2y^3, \{x, y\})$  is a proper cone extending both to the left and right direction. Figure 2 illustrates a collection of cones and a cone decomposition of a monomial ideal.

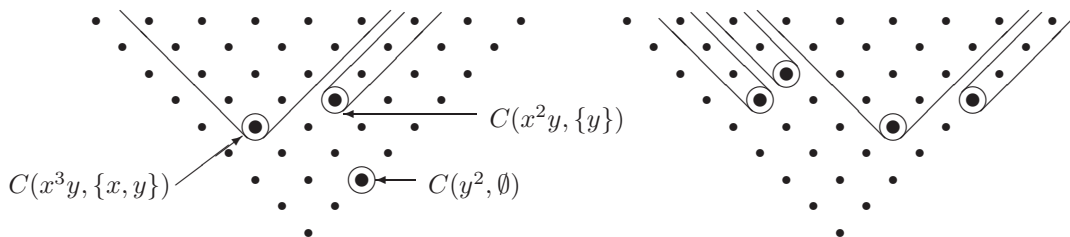


Figure 2: At the left we see three cones:  $C(y^2, \emptyset)$  is one dot,  $C(x^2y^3, \{y\})$  is a ray, and the cone  $C(x^3y, \{x, y\})$  has dimension 2. At the right we see the cone decomposition for  $I = \langle x^4y, xy^3, y^5 \rangle$  as the collection of cones  $\{C(x^4y, \{x\}), C(x^4y^2, \{x\}), C(xy^3, \{x, y\}), C(xy^3, \{x, y\}), C(y^5, \{y\})\}$ .

Let  $\{f_1, f_2, \dots, f_r\}$  be a set of homogeneous polynomials with  $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r\}$  a set of subsets of  $X$ . Then  $P = \{(f_1, \mathbf{u}_1), (f_2, \mathbf{u}_2), \dots, (f_r, \mathbf{u}_r)\}$  is a cone decomposition of  $R$ , a subring of  $\mathbb{K}[\mathbf{x}]$  if

$$R = C(f_1, \mathbf{u}_1) \oplus C(f_2, \mathbf{u}_2) \oplus \dots \oplus C(f_r, \mathbf{u}_r), \tag{8}$$

i.e.: the cones  $C(f_i, \mathbf{u}_i)$  form a direct decomposition of  $R$ . The right of Figure 2 shows a cone decomposition of a monomial ideal.

Lemma 5.1 provides an algorithm to split  $\mathbb{K}[\mathbf{x}]$  relative to an ideal  $I$ .

**Lemma 5.1** *Let  $I$  be a monomial ideal,  $f \in \mathbb{K}[\mathbf{x}]$ , and  $B$  is a basis for  $I : f$ . Then*

1.  $C(f, \mathbf{u}) \subseteq I \Leftrightarrow 1 \in B$ .
2.  $C(f, \mathbf{u}) \cap I = \emptyset \Leftrightarrow B \cap \{ \mathbf{u}^{\mathbf{a}} \mid \mathbf{a} \in \mathbb{N}^{|\mathbf{u}|} \}.$

*Proof.*

$$1. 1 \in B \Leftrightarrow 1 \in I : f \Leftrightarrow f \in I \Leftrightarrow C(f, X) \subseteq I.$$

2.  $\Rightarrow$  Assume  $C(f, \mathbf{u}) \cap I = \emptyset$ , then for any monomial  $\mathbf{u}^{\mathbf{a}}$ :

$$f\mathbf{u}^{\mathbf{a}} \in C(f, \mathbf{u}) \Rightarrow f\mathbf{u}^{\mathbf{a}} \notin I \quad (9)$$

$$\Rightarrow \mathbf{u}^{\mathbf{a}} \notin I : f \quad (10)$$

$$\Rightarrow \mathbf{u}^{\mathbf{a}} \notin B. \quad (11)$$

$\Leftarrow$  Assume  $B \cap \{ \mathbf{u}^{\mathbf{a}} \mid \mathbf{a} \in \mathbb{N}^{|\mathbf{u}|} \} = \emptyset$ . Then for any  $\mathbf{u}^{\mathbf{a}}$ :  $\mathbf{u}^{\mathbf{a}} \notin I : f$ , otherwise  $B$  would have to contain a divisor of  $\mathbf{u}^{\mathbf{a}}$  of the form  $\mathbf{u}^{\mathbf{b}}$ . So  $\mathbf{u}^{\mathbf{a}} \notin I : f \Rightarrow f\mathbf{u}^{\mathbf{a}} \notin I$ . By the definition of  $C(f, \mathbf{u})$ , every polynomial in  $C(f, \mathbf{u})$  is of the form  $f\mathbf{u}^{\mathbf{a}}$  and hence not in  $I$ .  $\square$

The Hilbert function of  $C(f, \mathbf{u})$  depends only on  $\deg(f)$  and the size  $|\mathbf{u}|$  of  $\mathbf{u}$ :

$$h_{C(f, \emptyset)}(s) = \begin{cases} 0 & \text{if } s \neq \deg(f) \\ 1 & \text{if } s = \deg(f) \end{cases} \quad (12)$$

and

$$\text{for } |\mathbf{u}| > 0 : h_{C(f, \mathbf{u})}(s) = \begin{cases} 0 & \text{if } s < \deg(f) \\ \binom{s - \deg(f) + |\mathbf{u}| - 1}{|\mathbf{u}| - 1} & \text{if } s \geq \deg(f). \end{cases} \quad (13)$$

For any cone decomposition  $P$  of  $R$ , the Hilbert function of  $R$  is

$$h_R(s) = \sum_{(f, \mathbf{u}) \in P} h_{C(f, \mathbf{u})}(s). \quad (14)$$

In [2], the following expression for the Hilbert polynomial of an ideal  $I$  is derived:

$$h_I(s) = \binom{s - d + n - 1}{n - 1} + \binom{s - d + n}{n} - 1 + \sum_{i=1}^n \binom{s - a_i + i - 1}{i}, \quad (15)$$

where the coefficients  $a_i$  are the coefficients of the Hilbert polynomial of the cone decomposition. Applying the backward difference operator and other techniques leads to bounds on the coefficients of the Hilbert polynomial.

## 6 Exercises

1. Use a computer algebra package to derive the Gregory-Newton interpolation formulas to interpolate  $(i, h_i) \in \mathbb{Z}^2$ , for  $i = d, d + 1, \dots, d + n$ . Show that the interpolating polynomial is a linear combination of binomial coefficients.
2. Explore the command `hilbertSeries` in Macaulay 2.

## References

- [1] D. Cox, J. Little, and D. O’Shea. *Ideals, Varieties and Algorithms. An Introduction to Computational Algebraic Geometry and Commutative Algebra*. Undergraduate Texts in Mathematics. Springer–Verlag, second edition, 1997.
- [2] T. Dubé. The structure of polynomial ideals and Gröbner bases. *SIAM J. Comput.*, 19(4):750–773, 1990.