Tropical Algebraic Geometry

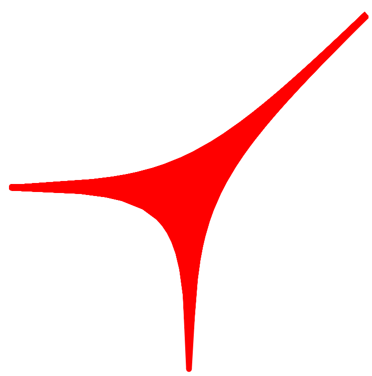
We follow [2] linking Newton polytopes to amoebas. Tropical algebraic geometry has its origins in the max-plus algebra. We state a generalization of Bernshteĭn's theorems.

1 An Asymptotic View on Varieties

Denoting $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, we consider the application of

$$\begin{array}{rcccc} \log & : & \mathbb{C}^* \times \mathbb{C}^* & \rightarrow & \mathbb{R} \times \mathbb{R} \\ & & (x, y) & \mapsto & (\log(|x|), \log(|y|)) \end{array} \quad (1)$$

to a variety. Following [2], $\log(V)$ for a variety V is called the amoeba of a variety. For a line, we use polar coordinates to plot the amoeba: $f := \frac{1}{2}x + \frac{1}{5}y - 1 = 0$, $A := \left[\ln\left(\left|re^{I\theta}\right|\right), \ln\left(\left|\frac{5}{2}re^{I\theta} - 5\right|\right)\right]$, see Figure 1.



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> f := 1/2*x + 1/5*y - 1:
> s := solve(f,y):
> L := map(log,map(abs,[x,s])):
> A := subs(x=r*exp(I*theta),L);
> Ap := seq(plot([op(subs(theta=k*Pi/200,A)),
        r=-100..100],thickness=6),k=0..99):
> plots[display](Ap,axes=none);
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Figure 1: The amoeba of a linear polynomial, with all Maple commands at the right.

We compactify the amoeba of $f^{-1}(0)$ by taking lines perpendicular to the tentacles. As each line cuts the plane in half, we keep those halves of the plane where the amoeba lives. The intersection of all half planes defines a polygon: the Newton polygon of f. For the amoeba in Figure 1, its compactification is shown in Figure 2. On Figure 2 we recognize the shape of the triangle, the Newton polygon of a linear polynomial.

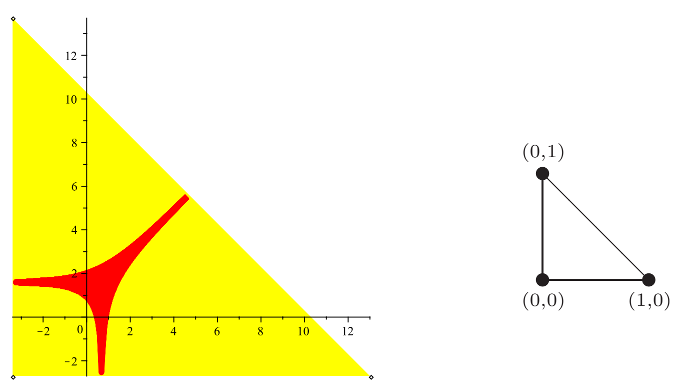

Figure 2: The compactification of the amoeba: the edges of the Newton polygon (displayed at the right) are perpendicular to the tentacles of the amoeba.

The tentacles of the amoeba stretch out to infinity and are represented by the inner normals, perpendicular to the edges of the Newton polygon. The inward pointing normal vectors to the edges represent the tentacles of the amoeba. The collection of inner normals to the edges of the Newton polygon forms *a tropicalization* of f, denoted by Trop(f). To formalize this notion, we introduce the following definitions.

Exponents and direction vectors are related through duality via the *inner product* is

$$\begin{array}{rccc} \langle \cdot, \cdot \rangle & : & \mathbb{Z}^2 \times \mathbb{Z}^2 & \rightarrow & \mathbb{Z} \\ & & ((i,j),(u,v)) & \mapsto & iu + jv. \end{array} \quad (2)$$

We arrive at a tropicalization of a polynomial via the normal fan to the Newton polygon P of the polynomial f. The *normal cone to a vertex* $\mathbf{p}$ *of* P is

$$\{ \ \mathbf{v} \in \mathbb{R}^2 \setminus \{0\} \mid \langle \mathbf{p}, \mathbf{v} \rangle = \min_{\mathbf{q} \in P} \langle \mathbf{q}, \mathbf{v} \rangle \ \}. \quad (3)$$

The *normal cone to an edge spanned by* $\mathbf{p}_1$ *and* $\mathbf{p}_2$ is

$$\{ \ \mathbf{v} \in \mathbb{R}^2 \setminus \{0\} \mid \langle \mathbf{p}_1, \mathbf{v} \rangle = \langle \mathbf{p}_2, \mathbf{v} \rangle = \min_{\mathbf{q} \in P} \langle \mathbf{q}, \mathbf{v} \rangle \ \}. \quad (4)$$

The *normal fan* of P is the collection of all normal cones to vertices and edges of P. Given f, *a tropicalization of* f, denoted by Trop(f), is a finite collection of inner normals (u,v) (normalized as $\gcd(u,v) = 1$) to the edges of the Newton polygon P of f.

Assigning degrees to the cones of the normal fan, we get tropical versions of the theorems of Bézout and Bernshteĭn, see [8, Chapter 9]. This tropical version is an interpretation of the computation of the mixed cells in a regular mixed-cell configuration via the common refinement of the normal fans to their edges.

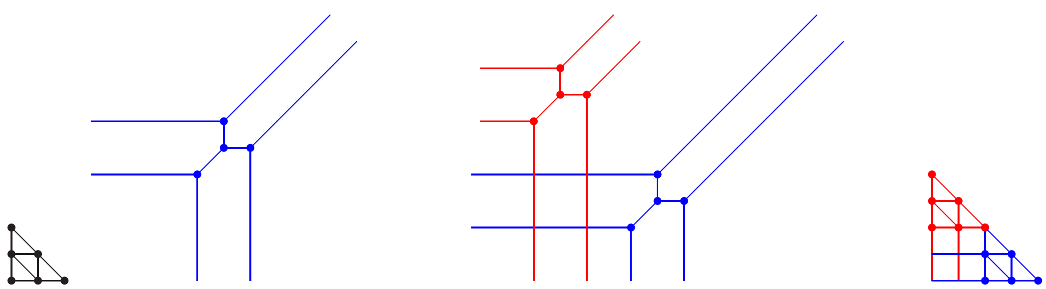

Figure 3: Tropical version of the theorem of Bézout: two general quadrics intersect in 4 points.

The intersections in Figure 3 correspond to inner normals for coordinate transformations in the method of Puiseux and in polyhedral homotopies. We now call such inner normals tropisms [6]. Table 1 shows an analogy between classical and tropical algebraic geometry, in the form of a dictionary.

classical	$\leftrightarrow$	tropical
$\mathbb{C}, +, \cdot$		$\mathbb{R}, \min, +$
polynomial		piecewise linear function
zero set		singular locus
hypersurface		normal fan
toric variety		ordinary linear variety
algebraic variety		balanced polyhedral fan

Table 1: A dictionary between classical and tropical algebraic geometry.

2 The Max-Plus Algebra at Work

Copying from the introduction of [3], consider in Figure 4 a schematic representation of two stations: S_1 and S_2; three circuits: one inner between the two stations and two outer circuits, serving the suburbs. Four trains travel back and forth.

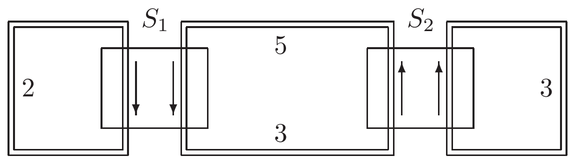

Figure 4: Two stations S_1 and S_2, three circuits, four trains. Numbers along the tracks are travel times.

The travel times along each track are fixed. A good timetable should obey the following rules:
(1) the frequency of trains should be as high as possible;
(2) the frequency of trains is the same along all tracks;
(3) trains wait to allow the changeover of passengers;
(4) the trains depart at a station as soon as allowed.

Denote by x_1 and x_2 the common departure times of the trains respectively at S_1 and S_2. The initial departure times are $x_1(0)$ and $x_2(0)$ and the kth departure times are given by $x_1(k-1)$ and $x_2(k-1)$. The rules above translate into the inequalities:

$$\begin{array}{c} x_1(k+1) = \max(x_1(k)+2, x_2(k)+5) \\ x_2(k+1) = \max(x_1(k)+3, x_2(k)+3) \end{array} \quad (5)$$

In the max-plus algebra, replacing max by $\oplus$ and addition by $\otimes$, we write

$$\begin{array}{c} x_1(k+1) = (x_1(k) \otimes 2) \oplus (x_2(k) \otimes 5) \\ x_2(k+1) = (x_1(k) \otimes 3) \oplus (x_2(k) \otimes 3) \end{array} \quad (6)$$

or in matrix-vector form:

$$\mathbf{x}(k+1) = A \otimes \mathbf{x}(k), \quad \mathbf{x}(k) = \left(\begin{array}{c} x_1(k) \\ x_2(k) \end{array} \right), \quad A = \left(\begin{array}{cc} 2 & 5 \\ 3 & 3 \end{array} \right). \quad (7)$$

Eigenvalues λ and eigenvectors $\mathbf{v}$ of a matrix A are defined in the max-plus algebra as

$$A \otimes \mathbf{v} = \lambda \otimes \mathbf{v}, \quad (8)$$

where not all components of $\mathbf{v}$ are equal to $-\infty$. Eigenvectors are (similar to conventional linear algebra) not uniquely determined. If the same constant is added to all components of an eigenvector, then we get again an eigenvector. It is typical to normalize an eigenvector so its first component is zero.

For the solution of the system above, we write

$$\mathbf{x}(1) = A \otimes \mathbf{x}(0) = \lambda \otimes \mathbf{x}(0) \quad (9)$$

and in general

$$\mathbf{x}(k) = \lambda^{\otimes k} \otimes \mathbf{x}(0), \quad k = 1, 2, \ldots \quad (10)$$

The max-plus algebra models phenomena in which the order of events is crucial. Instead of maxima, one can also work with a min-plus algebra. For our connections with Puiseux series (as we are usually interested in letting $t \to 0$), the min-plus algebra is often more convenient. Online resources are at `www.maxplus.org`.

3 The Fundamental Theorem

We start with some basic notations, needed to state the main theorem. The discrete valuation ring R_N of formal power series in $t^{1/N}$ for $N > 0$ is

$$R_N = \mathbb{K}[[t^{1/N}]] = \left\{ \sum_{\alpha=0}^{\infty} c_\alpha t^{\alpha/N} \mid c_\alpha \in \mathbb{K} \right\}, \tag{11}$$

with discrete valuation on $s \in R_N$:

$$\mathrm{val}(s) = \mathrm{ord}_t(s) = \min \left\{ \frac{\alpha}{N} \mid c_\alpha \neq 0 \right\} \in \frac{1}{N}\mathbb{Z} \cup \{\infty\}. \tag{12}$$

If N divides M, then R_N is a subring of R_M. Denote the quotient field of R_N as L_N. The direct limit of L_N gives the field of fractional power (or Puiseux) series

$$L = \mathbb{K}\{\{t\}\} = \varinjlim L_N = \bigcup_{N>0} L_N. \tag{13}$$

By the theorem of Puiseux, if $\mathbb{K}$ is algebraically closed, then so is $\mathbb{K}\{\{t\}\}$.

For an ideal J in the polynomial ring $\mathbb{K}\{\{t\}\}[\mathbf{x}]$, there are two ways to define the tropical variety of J, denoted by $\mathrm{Trop}(J)$:

1. using t-initial ideals:

$$\mathrm{Trop}(J) = \{ \ \omega \in \mathbb{R}^n \mid t - in_\omega(J) \text{ is monomial free } \}, \tag{14}$$

   where $t - in_\omega$ uses the weight $(-1, \omega)$ on monomials in $\mathbb{K}[t, \mathbf{x}]$; or

2. $\mathrm{Trop}(J)$ is the image of $-\mathrm{val}(p)$ for all $p \in V(J)$ of J in $\mathbb{K}\{\{t\}\}^n$.

The main theorem of tropical algebraic geometry states that both ways to defined tropical varieties are equivalent, or formally:

Theorem 3.1 (the fundamental theorem of tropical algebraic geometry)

$$\omega \in \mathrm{Trop}(J) \cap \mathbb{Q}^n \Leftrightarrow \exists p \in V(J) : -\mathrm{val}(p) = \omega \in \mathbb{Q}^n. \tag{15}$$

We can rephrase the theorem as follows: every rational vector in the tropical variety corresponds to the leading powers of a Puiseux series converging to a point in the variety. This theorem is a generalization of the theorem of Bernshteĭn, for which a constructive proof is given in [5].

A t-initial ideal of an ideal J is not generated by taking the t-initial forms of the given generators of J. Consider for example $J = \langle tx + y, x + t\rangle$ as an ideal in $\mathbb{K}\{\{t\}\}[x, y]$. We have that $y - t^2 \in J$. Let $\omega = (1, -1)$, so $y = t - in_\omega(y - t^2)$ belongs to the t-initial ideal. However $\langle t - in_\omega(tx + y), t - in_\omega(x + t)\rangle = \langle x\rangle$ and $y \not\in \langle x\rangle$. To compute t-initial ideals, standard bases are used.

For pure dimensional prime ideals, the situation is less complicated. The main theoretical result of [1] states

Theorem 3.2 *Let I and ideal in $\mathbb{C}\{\{t\}\}[\mathbf{x}]$. If $\sqrt{I}$ is prime of dimension d, then* $\mathrm{Trop}(I)$ *is a pure d-dimensional polyhedral fan.*

A finite intersection of tropical hypersurufaces is *a tropical prevariety*, defined in [7]. Every ideal I has a fine generating set $\{f_1, f_2, \ldots, f_r\}$ such that

$$\mathrm{Trop}(I) = \mathrm{Trop}(f_1) \cap \mathrm{Trop}(f_2) \cap \cdots \cap \mathrm{Trop}(f_r). \tag{16}$$

This set of generators is then called a tropical basis for the ideal. Every tropical variety is a tropical prevariety, but the converse does not hold. An algorithm to compute the tropical prevariety uses the common refinement of the polyhedral fans of hypersurfaces, using Gfan [4].

4 Exercises

1. Consider the amoeba of the product of two linear equations. How many tentacles to you see in each direction? For which choices of the coefficients do you see holes in the amoeba?

2. For ideals I and J show that $I \subset J \Rightarrow \mathrm{Trop}(I) \supset \mathrm{Trop}(J)$; $\mathrm{Trop}(I \cap J) = \mathrm{Trop}(I) \cup \mathrm{Trop}(J)$; and $\mathrm{Trop}(I + J) \subset \mathrm{Trop}(I) \cap \mathrm{Trop}(J)$.

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