

## Border Bases

Since most applications require numerical values for the solutions, the sensitivity of Gröbner basis to small changes of the input coefficients is often a drawback. Following [7], we see border bases as a more flexible alternative for Gröbner bases. Algebraically, according to [2], border bases describe an ideal from the outside. We follow [3] to connect oil fields with border bases.

### 1 Zero Dimensional Ideals

Given a polynomial system  $f(\mathbf{x}) = 0$ , we say that the ideal  $I = \langle f \rangle$  it generates is zero dimensional if its zero set  $V(I) = f^{-1}(\mathbf{0})$  consists of finitely many isolated points. Recall that for zero dimensional ideals, the dimension of the quotient ring  $R[I] = \mathbb{C}[\mathbf{x}]/I$  equals the cardinality of the solution set.

The quotient ring  $R[I]$  of an ideal  $I$  allows to calculate modulo  $I$ :

$$R[I] = \{ [p]_I \mid p \in \mathbb{C}[\mathbf{x}] \} \text{ is the residue class ring,} \quad (1)$$

with  $[p]_I$  the residue class of  $p$  mod  $I$ :

$$[p]_I = \{ r \in \mathbb{C}[\mathbf{x}] \mid p - r \in I \} \quad (2)$$

$$= \{ r \in \mathbb{C}[\mathbf{x}] \mid p(\mathbf{z}) = r(\mathbf{z}), \forall \mathbf{z} \in V(I) \}. \quad (3)$$

Consider for example

$$f(x, y) = \begin{cases} x^2 + 4xy + 4y^2 - 4 = 0 \\ 4x^2 - 4xy + y^2 - 4 = 0 \end{cases} \quad D = 2 \times 2. \quad (4)$$

According to Bézout's theorem, we expect four solutions. Row reduction on the coefficients leads to

$$\begin{cases} 15x^2 - 20xy - 12 = 0 \\ 20y^2 + 20xy - 12 = 0. \end{cases} \quad (5)$$

A natural basis for the quotient ring  $\mathbb{C}[x, y]/\langle f \rangle$  is  $\mathbf{b} = (1, x, y, xy)$ . Viewing the first equation in (5) to rewrite  $x^2$  and  $x^2y$  modulo the ideal leads to

$$x\mathbf{b} = x \begin{pmatrix} 1 \\ x \\ y \\ xy \end{pmatrix} = \begin{pmatrix} x \\ x^2 \\ xy \\ x^2y \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{12}{15} & 0 & 0 & \frac{20}{15} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{48}{125} & \frac{36}{125} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \\ xy \end{pmatrix} \quad (6)$$

Observe:  $A_x \mathbf{b} = x\mathbf{b}$  is an eigenvalue problem.

The multiplication matrices  $A_x$  and  $A_y$  have joint eigenvectors:  $A_x \mathbf{b} = x\mathbf{b}$  and  $A_y \mathbf{b} = y\mathbf{b}$ . This implies  $A_y A_x \mathbf{b} = x A_y \mathbf{b} = xy\mathbf{b}$  and  $A_x A_y \mathbf{b} = y A_x \mathbf{b} = yx\mathbf{b}$ , so  $A_x A_y = A_y A_x$ . The multiplication matrices form a commuting family of matrices.

The dual space of the quotient ring  $R[I]$ :

$$(R[I])^* = \{ l : R[I] \rightarrow \mathbb{C} : [p]_I \mapsto l(p) := l(r), r \in R[I], p - r \in I \}. \quad (7)$$

The dual of  $I$  is  $D[I] = \{ l : \mathbb{C}[\mathbf{x}] \rightarrow \mathbb{C} : l(p) = 0, \forall p \in I \}$ .

**Theorem 1.1**  $D[I] = (R[I])^*$  and  $R[I] = (D[I])^*$ .

For a vector space  $V$ , if  $\mathbf{b}$  is a basis for  $V$  and  $\mathbf{c}$  a basis for its dual  $V^*$ , then  $\mathbf{c}^T \mathbf{b}$  equals the identity matrix.

The ideal of the dual is  $I[D[I]] = \ker(D[I]) = \{ p \in \mathbb{C}[\mathbf{x}] : l(p) = 0, \forall l \in D[I] \} = I$ . Note that this relation is more satisfactory than  $I[Z[I]] = \sqrt{I}$ .

The dual defines the multiplicity structure of a multiple zero.

## 2 Exploring Oil Fields

A reservoir consists of three layers: water, oil, and gas, with water at the bottom and gas on top (because of their densities). Without a rock to cap the escape of gas, there can be no reservoir, so drilling is needed to extract the oil. Gas flows to the surface by itself and often the oil has enough pressure to come up as well. For the exploration, pressure levels are critical to recover as much oil as possible. In Figure 1 is a schematic representation of an oil field with two wells.

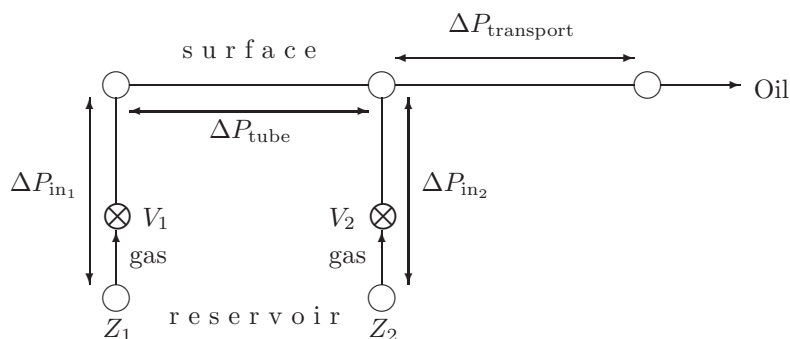


Figure 1: Schematic representation of a well with two zones  $Z_1$ ,  $Z_2$ , two pipes with valves  $V_1$ ,  $V_2$ . The empty circles are pressure readers. We observe four pressure differences:  $\Delta P_{in_1}$ ,  $\Delta P_{in_2}$ ,  $\Delta P_{tube}$ , and  $\Delta P_{transport}$ .

To link the gas production with pressure differences, we associate five variables to the physical quantities:  $x_1 = \Delta P_{in_1}$ ,  $x_2 = \Delta P_{in_2}$ ,  $x_3$  is the amount of gas produced,  $x_4 = \Delta P_{tube}$ , and  $x_5 = \Delta P_{transport}$ . These five quantities are assumed as the driving forces in the production of oil. The problem is to determine a polynomial  $f \in \mathbb{R}[x_1, x_2, \dots, x_5]$  based on measured values for the gas production, pressure differences, and the oil production. The result of the algebraic model is a polynomial  $f$  which explains the oil production in function of the driving forces.

A direct approach for this problem is to apply a least squares approximation on the given data, but this approach ignores the relations between the variables. Given is a set of approximate data points  $\mathbb{X} \subset \mathbb{R}^5$ , normalized properly so  $\mathbb{X} \subset [-1, +1]^5$ . For exact points (or if we would map  $\mathbb{X}$  into  $\mathbb{Q}$ ), the Buchberger-Möller algorithm produces a Gröbner basis  $G$  for the vanishing ideal. Using the division algorithm to write any polynomial  $p$  as

$$p = \sum_{g \in G} q_g g + f, \quad q_g \in \mathbb{Q}[\mathbf{x}] \quad (8)$$

means that we take into account the relations between the variables, relations imposed by the given data points. As  $g \in G$  vanishes at any point, the interpolating polynomial has its shape determined by the monomials in the remainder  $f$ . In particular, the polynomial in our algebraic model has the form

$$f = \sum_{\mathbf{x}^a \notin \text{in}_>(G)} c_a \mathbf{x}^a, \quad c_a \in \mathbb{R}. \quad (9)$$

The problem is then to find a suitable monomial basis for  $f$ .

While the problem with Gröbner bases for approximate data is their sensitivity to small perturbations, we end by pointing out that the order of the variables still matters. Calculations reported in [3] with degree reverse lexicographical term order result in a polynomial with a dominant presence of  $x_5$ , the last variable. This dominance of  $x_5$  in the model is unfortunate as  $x_5$  is the pressure difference in the transport pipe on the surface and most unlikely to be a key factor. Therefore, a relabeling of the variables according to the physical hierarchy leads to an interpolating polynomial more consistent with the physical interpretation.

### 3 An ill-conditioned Gröbner basis

As application area for this lecture we consider Gröbner bases that are very sensitive to small perturbations. The example is attributed to Windsteiger in [7, Example 4.2, page 103].

$$f(x, y) = \begin{cases} f_1 = -4 + 3 \left( \frac{172966043}{174178537}x - \frac{42176556}{358072327}y \right)^2 + \left( \frac{1}{3} + \frac{42176556}{358072327}x + \frac{172966043}{174178537}y \right)^2 = 0 \\ f_2 = -4 + \left( \frac{1}{3} - \frac{42176556}{358072327}y + \frac{172966043}{174178537}x \right)^2 + 4 \left( \frac{172966043}{174178537}y + \frac{42176556}{358072327}x \right)^2 = 0. \end{cases} \quad (10)$$

After the application of `evalf` and `expand` in Maple:

$$\tilde{f}(x, y) = \begin{cases} -3.888888889 + 2.972252063x^2 - 0.4678714642xy + 1.027747937y^2 \\ \quad + 0.07852520812x + 0.6620258576y = 0 \\ -3.888888889 - 0.07852520812y + 0.6620258576x + 3.958378094y^2 \\ \quad + 0.7018071964xy + 1.041621906x^2 = 0. \end{cases} \quad (11)$$

The coefficients of the system are well scaled. Looking at the geometry of the solution sets of the two equations, we see that the intersections occur at places where the curves meet. Small changes in the curves will lead to small shifts in the intersection points, see Figure 2.

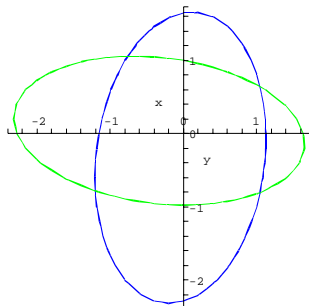


Figure 2: The example of Windsteiger.

If we compute a Gröbner basis eliminating  $y$ , then we see that the  $x$ -values of the solutions are clustered. Instead of taking the basis  $(1, x, x^2, x^3)$ , the basis  $(1, x, x^2, y)$  gives better results.

### 4 Border Bases

Given an ideal  $I$ , a normal set  $N[I]$  is a set of monomials  $N[I] = \{ \mathbf{x}^{\mathbf{a}} \mid \mathbf{a} \in A \}$  such that

1. for all  $\mathbf{x}^{\mathbf{a}} \in N[I]$ : if  $\mathbf{x}^{\mathbf{a}'}$  divides  $\mathbf{x}^{\mathbf{a}}$ , then  $\mathbf{x}^{\mathbf{a}'} \in N[I]$ ; and
2.  $B = \{ [\mathbf{x}^{\mathbf{a}}]_I \mid \mathbf{x}^{\mathbf{a}} \in N[I] \}$  is a basis for  $R[I]$ .

The normal set and the multiplication matrices provide a normal form for every polynomial modulo the ideal, generalizing the division algorithm for univariate polynomials. Given a normal set  $N$ , the border set is  $B[N] = \{ \mathbf{x}^{\mathbf{a}} \mid \mathbf{x}^{\mathbf{a}}/x_k \in N, \text{ for some } k \}$ . The border set collects all leading monomials in a border basis, defined by  $A_{x_k} \mathbf{b} = x_k \mathbf{b}$ , for all  $k = 1, 2, \dots, n$ , and where  $\mathbf{b}$  is  $N_{<}$ , the ordered normal set.

The existence of a border basis is implied by a Gröbner basis, i.e.: given a Gröbner basis for a zero dimensional ideal, we can directly write down a border basis.

## 5 Computing Border Bases

Methods to compute border bases are discussed in [1], [3], [4, 6], and [8], briefly summarized below.

### 5.1 the mixed volume equals the size of the border basis

In [8], the direct computation of border basis starts with the calculation of the mixed volume (also known as the BKK bound). The mixed volume of course is restricted to systems with as many equations as unknowns, but this restriction makes sense if we consider empirical polynomials on input. If the input polynomials have coefficients subject to roundoff, then we expect to find only isolated roots anyway. For practical purposes, good software exists to compute mixed volumes fast, in general faster than running the polyhedral homotopies.

Once the size of the normal set is determined via the mixed volume, the approach sketched in [8] starts by filling in the normal set with the lowest degree monomials that appear in the system. Via “autoreduction” we rewrite products of monomials with some variable modulo the equations in the given system. For numerical stability, this autoreduction is not bound to one particular fixed term order so we can use pivots as in the row reduction for linear systems. This pivoting is usually prevented by the term order imposed in advance and causes direct numerical implementations of Buchberger’s algorithm to fail.

### 5.2 algebraic algorithms

In [1], several algorithms are presented to compute border bases for zero dimensional ideals. The authors of [1] define a border prebasis as a collection of polynomials of the form

$$g_j = b_j - \sum_{t \in \mathcal{N}} c_{jt} t, \quad c_{ij} \in \mathbb{C}, j = 1, 2, \dots, s, \quad (12)$$

where  $\mathcal{N}$  is a normal set for the ideal. Two polynomials  $g_k$  and  $g_l$  in a border prebasis are neighbors if their  $S$ -polynomial equals  $S(g_k, g_l) = x_i g_k - x_j g_l$  for some variables  $x_i$  and  $x_j$ .

In analogy with the Buchberger criterion for a set of polynomials to be a Gröbner basis, there is a Buchberger criterion for a border bases, formalized in the next proposition (from [1]).

**Proposition 5.1 (Buchberger criterion for border basis)** *Let  $G$  be a border prebasis of a zero dimensional ideal  $I$ .  $G$  is a border basis for  $I$  if and only if for each pair of neighboring prebasis polynomials  $g_i$  and  $g_j$  of  $G$  there are coefficients  $c_g$  such that*

$$S(g_i, g_j) = \sum_{g \in G} c_g g. \quad (13)$$

Relationships between generators of an ideal are called syzygies. A special type of syzygies is the  $S$ -polynomial, designed to cancel leading terms. As we relax the term orders, the algorithm now computes syzygies. With a normal form algorithm to reduce syzygies with respect to the current border prebasis we have a Buchberger type algorithm for computing border bases. Switching term orders is similar to the FGLM algorithm and in [1] an example is given which shows an improvement of the Buchberger algorithm.

### 5.3 stable normal forms

In [6], numerical results are given for classical benchmark polynomial systems. For numerical stability of the border bases, Newton-like algorithms are suggested. The authors of [6] make an interesting mathematical connections to prolongations used in differential algebra. Their algorithms are implemented in SYNAPS [5].

## 5.4 approximate ideals and SVD

In [3], an  $\epsilon$ -approximate vanishing ideal  $I$  of a set of points  $\mathbb{X}$  is a system of generators  $f_i$ ,  $i = 1, 2, \dots, N$  of  $I$  such that  $\|f_i\| = 1$  for all  $i$  and  $|f_i(\mathbf{z})| < \epsilon$  for all  $\mathbf{z} \in \mathbb{X}$ .

A set  $G$  is an  $\epsilon$ -approximate border basis for the ideal  $I$  generated by  $G$  if for all  $g_i, g_j \in G$ : the  $S$ -polynomial  $|S(g_i, g_j)| < \epsilon$ .

The singular value decomposition (SVD) is proposed in [3] to compute the kernel of the monomials in the normal set, evaluated at the points in  $\mathbb{X}$ . The normal set is constructed incrementally. Applying row reduction with pivoting the algorithm constructs polynomials that vanish  $\epsilon$ -approximately on  $\mathbb{X}$ . The algorithm is the numerical analogue to the Buchberger-Möller algorithm.

## 6 Exercises

1. Compute a lexicographic Gröbner basis for (10), use it to compute the roots. Compare the results of the eliminating orders  $x > y$  and  $y > x$ .
2. Apply the Maple commands `Groebner[SetBasis]` and `Groebner[MulMatrix]` to a Gröbner basis for (10).
3. Download and install SYNAPS [5] on your computer and compute some examples of border bases. Compare the running time with the computation of a Gröbner basis in any computer algebra system.

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