Binomial Ideals

Binomial ideals offer an interesting class of examples. Because they occur so frequently in various applications, the development methods for binomial ideals is justified.

1 Binomial Ideals

Consider $\mathbb{K}[\mathbf{x}]$, with $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$. Typically we will assume that $\mathbb{K}$ is algebraically closed, so $\mathbb{K} = \mathbb{C}$ is our default coefficient field. Then

$$I = \left\langle\ c_{\mathbf{a}} x^{\mathbf{a}} - c_{\mathbf{b}} x^{\mathbf{b}} \mid \mathbf{a}, \mathbf{b} \in \mathbb{N}^n, c_{\mathbf{a}}, c_{\mathbf{b}} \in \mathbb{K}^*\ \right\rangle \tag{1}$$

is a binomial ideal. A polynomial is a binomial if it has exactly two monomials with a nonzero coefficient. A binomial ideal is generated by binomials. If the generators are of the form $\mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{b}}$ (all coefficients $c_{\mathbf{a}}$ and $c_{\mathbf{b}}$ are one), then the binomial ideal is a toric ideal. Because the exponents determine the structure of the ideal, we then define a toric ideal as

$$I_A = \left\langle\ \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \mid \mathbf{u}, \mathbf{v} \in \mathbb{N}^n \text{ and } A\mathbf{u} = A\mathbf{v}\ \right\rangle. \tag{2}$$

The solution set of a toric ideal is a toric variety — see [6] for a short and [1] for a longer definition. As an alternative to the ideal description, a toric variety over $\mathbb{C}$ is defined as a complex algebraic variety with an action of $(\mathbb{C}^*)^n$ and a dense open subset isomorphic to $(\mathbb{C}^*)^n$ carrying the regular action, i.e.: a toric variety is an algebraic torus closure.

The efficient manipulation of monomial ideals requires combinatorics [7]. Via the introduction of new variables we can rewrite any system of polynomials into a system of trinomials, of polynomials with three monomials (or less). This trick does not allow the reduction of any system to a binomial system. Binomial ideals have special properties, for instance:

Theorem 1.1 (Theorem 2.6 in [2]) *If $\mathbb{K}$ is algebraically closed and I is a binomial ideal in $\mathbb{K}[\mathbf{x}]$, then every associated prime of I is generated by binomials.*

The condition that $\mathbb{K}$ is algebraically closed is essential, as the following example over $\mathbb{Q}$ shows: $\langle x^3 - 1 \rangle = \langle x - 1 \rangle \cap \langle x^2 + x + 1 \rangle$. If we extend $\mathbb{Q}$ with $w = e^{(2\pi\sqrt{-1})/3}$, then over $\mathbb{Q}(w)$ the binomial ideal factors as $\langle x^3 - 1 \rangle = \langle x - 1 \rangle \cap \langle x + (1 - \sqrt{-3})/2 \rangle \cap \langle x + (1 + \sqrt{-3})/2 \rangle$.

The frequent occurring of binomial ideals in applications justifies the development of specific methods to solve binomial systems.

In polyhedral homotopies we encounter the special case where the number of generators equals n and only solutions in $(\mathbb{C}^*)^n$ are interesting. After writing the system in a normal form as $\mathbf{x}^A = \mathbf{c}$, with $A \in \mathbb{Z}^{n \times n}$ and $\mathbf{c} \in (\mathbb{C}^*)^n$, we conclude that there are exactly as many regular solutions as $|\det(A)|$.

2 Hidden Markov Models

Not all problems In algebraic statistics [3] lead to binomial ideals, as the application of hidden Markov models and the maximum likelihood equations shows.

The following example is taken from [8]. Consider a sequence of letters over the four letter alphabet $\{A, C, G, T\}$. Suppose the sequence was generated rolling four tetrahedral dice, where the probabilities of rolling each letter are known:

	A	C	G	T
first die	0.15	0.33	0.36	0.16
second die	0.27	0.24	0.23	0.26
third die	0.25	0.25	0.25	0.25

(3)

The probabilities of picking which dice is unknown.

If θ_1 is the probability of using the first die and θ_2 the probability for the second die, then the probability that the third die is picked equals $1 - \theta_1 - \theta_2$.

The probabilities of each letter showing up in the sequence are then functions of θ_1 and θ_2. Denote these functions by p_A, p_C, p_G, and p_T. The likelihood $L(\theta_1, \theta_2)$ of observing any given sequence is then the product of $p_A^{n_A} p_C^{n_C} p_G^{n_G} p_T^{n_T}$, where n_A, n_C, n_G, and n_T equal the number of times the respective letter A, C, G, and T occur in the sequence.

The statistical principles of maximal likelihood presumes that those parameter values for θ_1 and θ_2 were used which make the likelihood of observing the sequence as large as possible. Maximizing $L(\theta_1, \theta_2)$ is equivalent to maximizing $\ell(\theta_1, \theta_2) = \log(L(\theta_1, \theta_2))$. A straightforward solution to this optimization problem is to search for the critical points, and solving the system defined by the rational functions $\frac{\partial \ell}{\partial \theta_1} = 0$ and $\frac{\partial \ell}{\partial \theta_2} = 0$. As $\ell(\theta_1, \theta_2) = n_A \log(p_A) + n_C \log(p_C) + n_G \log(p_G) + n_T \log(p_T)$, the system is

$$\left\{ \begin{array}{l} \frac{\partial \ell}{\partial \theta_1} = \frac{n_A}{p_A}\frac{\partial p_A}{\partial \theta_1} + \frac{n_C}{p_C}\frac{\partial p_C}{\partial \theta_1} + \frac{n_G}{p_G}\frac{\partial p_G}{\partial \theta_1} + \frac{n_T}{p_T}\frac{\partial p_T}{\partial \theta_1} = 0 \\ \frac{\partial \ell}{\partial \theta_2} = \frac{n_A}{p_A}\frac{\partial p_A}{\partial \theta_2} + \frac{n_C}{p_C}\frac{\partial p_C}{\partial \theta_2} + \frac{n_G}{p_G}\frac{\partial p_G}{\partial \theta_2} + \frac{n_T}{p_T}\frac{\partial p_T}{\partial \theta_2} = 0. \end{array} \right. \quad (4)$$

Of course, only those solutions which lie in the interval $[0, 1]$ are interesting.

A hidden Markov model is a polynomial map from the parameter space to the probability space. Via maximum likelihood, this hidden model is reconstructed. Another nice example is that of an occasionally dishonest casino, discussed in [8].

As formulated as polynomial systems, the constraint optimization problem has a specific number of solutions, the so-called maximum likehood degree, derived in [4].

3 Cellular decompositions

Following [5], consider $\mathcal{E} \subseteq \{1, 2, \ldots, n\}$ and denote the algebraic torus corresponding to $\mathcal{E}$ by

$$(\mathbb{K}^*)^{\mathcal{E}} = \{\ \mathbf{x} \in \mathbb{K}^n \mid x_i \neq 0 \text{ for } i \in \mathcal{E} \text{ and } x_j = 0 \text{ for } j \notin \mathcal{E}\ \}. \quad (5)$$

The central definition is

Definition 3.1 A proper binomial ideal I in $\mathbb{K}[\mathbf{x}]$ is *cellular* if each variable x_i is either a nonzerodivisor or nilpotent modulo I.

Primary ideals I are cellular as every element in $\mathbb{K}[\mathbf{x}]/I$ is either nilpotent or a nonzerodivisor.

We have a characterization for an ideal I being cellular in the following lemma.

Lemma 3.1 *A proper binomial ideal I in $\mathbb{K}[\mathbf{x}]$ is cellular if and only if there exists a set $\mathcal{E} \subseteq \{1, 2, \ldots, n\}$ of indices of variables in $\mathbf{x}$ such that*

1. $I = \left(I : \left(\prod_{i \in \mathcal{E}} x_i \right)^\infty \right)$*; and*
2. *For every $i \notin \mathcal{E}$, there exists an integer $d_i \geq 0$ such that $\langle x_i^{d_i} \mid i \notin \mathcal{E} \rangle$ is contained in I.*

The `Binomials` package in Macaulay 2 provides an implementation of the following recursive algorithm:

Algorithm 3.1 (cellular decomposition)
Input: a binomial ideal I.
Output: a cellular decomposition of I.

1. If I is cellular, then return I.
2. Choose x_i that is a zerodivisor but not nilpotent modulo I.
3. Determine the power m such that $(I : x_i^m) = (I : x_i^\infty)$.
4. Call the algorithm on $(I : x_i^m)$ and $I + \langle x_i^m \rangle$.

The algorithm to compute a cellular decomposition is the first step in the following algorithm to solve toric ideals.

Algorithm 3.2 (solve toric ideals)
Input: a zero dimensional toric ideal I.
Output: roots of unity to extend $\mathbb{Q}$ and solutions in $V(I)$.

1. Compute a cellular decomposition of I.
2. For each cellular component do
2.1 Set the noncell variables to zero and determine the product $D := \prod_{i \notin \mathcal{E}} d_i$
   of the minimal powers of the noncell variables.
2.2 Compute a lexicographic Gröbner basis and
   solve the lattice ideal of the cellular component, adjoining roots of unity.
2.3. Save each solution D times.
3. Compute the least common multiple m of the powers of the adjoined roots of unity
   and construct the cyclotomic field $\mathbb{Q}(w_m)$.
4. Return the list of solutions as elements in $\mathbb{Q}(w_m)$.

4 Using Macaulay 2

The package `Binomials` of Thomas Kahle [5] is available in Macaulay 2, since version 1.4. We run some examples of [5] below.

```
i1 : S = QQ[x,y,z];
i2 : I = ideal(x^2-y,y^3-z,x*y-z);
i3 : loadPackage "Binomials";
i4 : binomialSolve I
BinomialSolve created a cyclotomic field of order 3
```

```
o4 = {{1, 1, 1}, {- ww  - 1, ww , 1},
                      3       3
    {ww , - ww  - 1, 1},
       3      3
    {0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
i5 : degree I
o5 = 6
```

We continue the session computing a binomial primary decomposition:

```
i6 : BPD I
Running cellular decomposition:
cellular components found: 1
cellular components found: 2
Decomposing cellular components:
Decomposing cellular component: 1 of 2
1 monomial to consider for this cellular component
BinomialSolve created a cyclotomic field of order 3
done
Decomposing cellular component: 2 of 2
3 monomials to consider for this cellular component
done
Removing redundant components...
4 Ideals to check
3 Ideals to check
2 Ideals to check
1 Ideals to check
0 redundant ideals removed. Computing mingens of result.

o6 = {ideal (z - 1, y - 1, x - 1),

   ideal (z - 1, y - ww , x + ww  + 1),
                       3       3

   ideal(z - 1, y + ww  + 1, x - ww ),
                      3            3

             2        2
   ideal (z, y , x*y, x  - y)}
```

We consider the last ideal in the primary decomposition

```
i7 : I = ideal(z,y^2,x*y,x^2 - y);
i8 : binomialAssociatedPrimes I
3 monomials to consider for this cellular component

o8 = {ideal (z, y, x)}
```

Cellular decompositions are computed as follows:

```
i1 : loadPackage "Binomials";
i2 : S = QQ[x1,x2,x3,x4,x5];
i3 : I = ideal(x1*x4^2-x2*x5^2, x1^3*x3^3-x2^4*x4^2, x2*x4^8-x3^3*x5^6);
```

```
i4 : I
                2        2    3  3     4  2       8     3  6
o4 = ideal (x1*x4  - x2*x5 , x1 x3  - x2 x4 , x2*x4  - x3 x5 )
i5 : BCD I
cellular components found: 1
redundant component
redundant component
cellular components found: 2

                 2        2    3  3     4  2    3  4     2  3  2    2  6
o5 = {ideal (x1*x4  - x2*x5 , x1 x3  - x2 x4 , x2 x4  - x1 x3 x5 , x2 x4  -

         3  4       8     3  6
     x1*x3 x5 , x2*x4  - x3 x5 ),

              2       2       2    5    6    4  2    8
     ideal (x1 , x1*x4  - x2*x5 , x2 , x5 , x2 x4 , x4 )}
```

5 Assignments

1. Install version 1.4 of Macaulay 2 and explore the package Binomials.
2. Explore the capabilities in CoCoA for handling binomial ideals.

References

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