Tropical Algebraic Geometry

We follow [1] linking Newton polytopes to amoebas. Tropical algebraic geometry has its origins in the max-plus algebra. An introduction to tropical geometry appears in [4].

1 An Asymptotic View on Varieties

Denoting $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, we consider the application of

$$\log : \mathbb{C}^* \times \mathbb{C}^* \rightarrow \mathbb{R} \times \mathbb{R}$$

$$(x, y) \mapsto (\log(|x|), \log(|y|))$$

(1)

to a variety. Following [1], $\log(V)$ for a variety $V$ is called the amoeba of a variety. For a line, we use polar coordinates to plot the amoeba: $f := \frac{1}{2}x + \frac{1}{5}y - 1 = 0$, $A := [\ln (|re^{i\theta}|), \ln ((\frac{5}{2}re^{i\theta} - 5))]$, see Figure 1.

Figure 1: The amoeba of a linear polynomial, with all Maple commands at the right.

We compactify the amoeba of $f^{-1}(0)$ by taking lines perpendicular to the tentacles. As each line cuts the plane in half, we keep those halves of the plane where the amoeba lives. The intersection of all half planes defines a polygon: the Newton polygon of $f$. For the amoeba in Figure 1, its compactification is shown in Figure 2. On Figure 2 we recognize the shape of the triangle, the Newton polygon of a linear polynomial.

Figure 2: The compactification of the amoeba: the edges of the Newton polygon (displayed at the right) are perpendicular to the tentacles of the amoeba.
The tentacles of the amoeba stretch out to infinity and are represented by the inner normals, perpendicular to the edges of the Newton polygon. The inward pointing normal vectors to the edges represent the tentacles of the amoeba. The collection of inner normals to the edges of the Newton polygon forms a tropicalization of \( f \), denoted by \( \text{Trop}(f) \). To formalize this notion, we introduce the following definitions.

Exponents and direction vectors are related through duality via the inner product

\[
\langle \cdot, \cdot \rangle : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{Z} \quad \langle (i, j), (u, v) \rangle \mapsto iu + jv.
\]

We arrive at a tropicalization of a polynomial via the normal fan to the Newton polygon \( P \) of the polynomial \( f \). The normal cone to a vertex \( p \) of \( P \) is

\[
\{ v \in \mathbb{R}^2 \setminus \{0\} \mid \langle p, v \rangle = \min_{q \in P} \langle q, v \rangle \}.
\]

The normal cone to an edge spanned by \( p_1 \) and \( p_2 \) is

\[
\{ v \in \mathbb{R}^2 \setminus \{0\} \mid \langle p_1, v \rangle = \langle p_2, v \rangle = \min_{q \in P} \langle q, v \rangle \}.
\]

The normal fan of \( P \) is the collection of all normal cones to vertices and edges of \( P \). Given \( f \), a tropicalization of \( f \), denoted by \( \text{Trop}(f) \), is a finite collection of inner normals \( (u, v) \) (normalized as \( \gcd(u, v) = 1 \)) to the edges of the Newton polygon \( P \) of \( f \).

Assigning degrees to the cones of the normal fan, we get tropical versions of the theorems of Bézout and Bernstein, see [7, Chapter 9]. This tropical version is an interpretation of the computation of the mixed cells in a regular mixed-cell configuration via the common refinement of the normal fans to their edges.

![Figure 3: Tropical version of the theorem of Bézout: two general quadrics intersect in 4 points.](image)

The intersections in Figure 3 correspond to inner normals for coordinate transformations in the method of Puiseux and in polyhedral homotopies. We now call such inner normals tropisms [5]. Table 1 shows an analogy between classical and tropical algebraic geometry, in the form of a dictionary.

<table>
<thead>
<tr>
<th>classical</th>
<th>tropical</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{C}, +, \cdot )</td>
<td>( \mathbb{R}, \min, + )</td>
</tr>
<tr>
<td>polynomial</td>
<td>piecewise linear function</td>
</tr>
<tr>
<td>zero set</td>
<td>singular locus</td>
</tr>
<tr>
<td>hypersurface</td>
<td>normal fan</td>
</tr>
<tr>
<td>toric variety</td>
<td>ordinary linear variety</td>
</tr>
<tr>
<td>irreducible algebraic variety</td>
<td>balanced ((n - 1))-skeleton</td>
</tr>
</tbody>
</table>

Table 1: A dictionary between classical and tropical algebraic geometry.
2 The Max-Plus Algebra at Work

Copying from the introduction of [2], consider in Figure 4 a schematic representation of two stations: $S_1$ and $S_2$; three circuits: one inner between the two stations and two outer circuits, serving the suburbs. Four trains travel back and forth.

![Figure 4: Two stations $S_1$ and $S_2$, three circuits, four trains. Numbers along the tracks are travel times.](image)

The travel times along each track are fixed. A good timetable should obey the following rules:

1. The frequency of trains should be as high as possible;
2. Frequency of trains is the same along all tracks;
3. Trains wait to allow the changeover of passengers;
4. The trains depart at a station as soon as allowed.

Denote by $x_1$ and $x_2$ the common departure times of the trains respectively at $S_1$ and $S_2$. The initial departure times are $x_1(0)$ and $x_2(0)$ and the $k$th departure times are given by $x_1(k-1)$ and $x_2(k-1)$. The rules above translate into the inequalities:

$$
x_1(k + 1) = \max(x_1(k) + 2, x_2(k) + 5)
$$
$$
x_2(k + 1) = \max(x_1(k) + 3, x_2(k) + 3)
$$

In the max-plus algebra, replacing max by $\oplus$ and addition by $\otimes$, we write

$$
x_1(k + 1) = (x_1(k) \otimes 2) \oplus (x_2(k) \otimes 5)
$$
$$
x_2(k + 1) = (x_1(k) \otimes 3) \oplus (x_2(k) \otimes 3)
$$

or in matrix-vector form:

$$
x(k + 1) = A \otimes x(k), \quad x(k) = \begin{pmatrix}
x_1(k) \\
x_2(k)
\end{pmatrix}, \quad A = \begin{pmatrix}
2 & 5 \\
3 & 3
\end{pmatrix}.
$$

Eigenvalues $\lambda$ and eigenvectors $v$ of a matrix $A$ are defined in the max-plus algebra as

$$
A \otimes v = \lambda \otimes v,
$$

where not all components of $v$ are equal to $-\infty$. Eigenvectors are (similar to conventional linear algebra) not uniquely determined. If the same constant is added to all components of an eigenvector, then we get again an eigenvector. It is typical to normalize an eigenvector so its first component is zero.

For the solution of the system above, we write

$$
x(1) = A \otimes x(0) = \lambda \otimes x(0)
$$

and in general

$$
x(k) = \lambda^k \otimes x(0), \quad k = 1, 2, \ldots
$$

The max-plus algebra models phenomena in which the order of events is crucial. Instead of maxima, one can also work with a min-plus algebra. For our connections with Puiseux series (as we are usually interested in letting $t \to 0$), the min-plus algebra is often more convenient. Online resources are at [www.maxplus.org](http://www.maxplus.org).
3 Polyhedral Methods for Algebraic Sets

This section is based on [8]. To compute all isolated solutions of a polynomial system with polyhedral methods, we apply polyhedral homotopies. Tropical algebraic geometry provides us a framework to generalized polyhedral homotopies to positive dimensional solution sets.

3.1 solving the cyclic 4-roots problem

Algebraic curves defined by binomial systems can be solved by one tropism but it may happen that one tropism solves a more general system. Consider for example the cyclic 4-roots problem.

$$f(x) = \begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 = 0 \\ x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 = 0 \\ x_1x_2x_3x_4 - 1 = 0 \end{cases} \tag{11}$$

There is one tropism $v = (1, -1, 1, -1)$ which leads to the initial form system $\text{in}_v f(z) = 0$:

$$\text{in}_v f(x) = \begin{cases} x_2 + x_4 = 0 \\ x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 = 0 \\ x_2x_3 + x_1x_2x_3 - 1 = 0 \end{cases} \tag{12}$$

The system $\text{in}_v f(y) = 0$ has two solutions and we find the two solution curves: $(t, -t^{-1}, -t, -t^{-1})$ and $(t, t^{-1}, -t, -t^{-1})$.

Note that $v = (-1, 1, -1, 1)$ is a tropism as well for the solution curves of cyclic 4-roots, but considering this tropism corresponds to setting $x_1 = t^{-1}$ or moving the curve to infinity instead of to zero as $t$ goes to zero.

3.2 Asymptotics of Witness Sets

One way to compute tropisms would be to start from a witness set for an algebraic curve in $n$-space given by $d$ points on a general hyperplane $c_0 + c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0$ and satisfying a system $f(x) = 0$. We then deform a witness set for a curve in two stages:

1. The first homotopy moves to a hyperplane in special position:

$$h(x, t) = \begin{cases} f(x) = 0 \\ (c_0 + c_1x_1 + \cdots + c_nx_n)t + (c_0 + c_1x_1)(1 - t) = 0, \text{ for } t \text{ from } 1 \text{ to } 0. \end{cases} \tag{13}$$

2. After renaming $c_0 + c_1x_1 = 0$ into $x_1 = \gamma$, we let $x_1$ go to zero with the following homotopy:

$$h(x, t) = \begin{cases} f(x) = 0 \\ x_1 - \gamma t = 0, \text{ for } t \text{ from } 1 \text{ to } 0. \end{cases} \tag{14}$$

The two homotopies need further study. In the first homotopy (13) some paths will diverge, consider for example $f(x_1, x_2) = x_1x_2 - 1$. Even all paths may diverge if the solution curve lies in some hyperplane perpendicular to the first coordinate axis $x_1 = c$ with $c$ different from $-c_0/c_1$.

**Lemma 3.1** All solutions at the end of the homotopy $h(x, t) = 0$ of (13) lie on the curve defined by $f(x) = 0$ and in the hyperplane $x_1 = -c_0/c_1$. 
Proof. We claim that we find the same solutions to \( h(x, t = 0) = 0 \) either by using the homotopy in (13) or by solving \( h(x, 0) = 0 \) directly. This claim follows from cheater’s homotopy [3] or the more general coefficient-parameter polynomial continuation [6]. \( \square \)

The second claim we make in the main theorem below is that we recover all data lost with tropisms. For simplest example of the hyperbola \( x_1 x_2 - 1 = 0 \): its solution is \((x_1 = t, x_2 = t^{-1})\) and the tropism is \( \nu = (1, -1) \). The lemma below extends the normal form for the Puiseux series expansion for plane curves (as used in [9]) to general space curves.

**Lemma 3.2** As \( t \to 0 \) in the homotopy (14), the leading powers of the Puiseux series expansions are the components of a tropism. In particular, the expansions have the form

\[
\begin{align*}
  x_1 &= t \\
  x_k &= c_k t^{v_k} (1 + O(t)), \quad k = 2, \ldots, n.
\end{align*}
\]

**Proof.** Following Bernstein’s second theorem, a solution at infinity is a solution in \((\mathbb{C}^*)^n\) of an initial form system. For a solution to have values in \((\mathbb{C}^*)^n\), all equations in that system need to have at least two monomials. So the system is an initial form system defined by a tropism. To arrive at the form of (15) for the solution defined by the homotopy (14) we rescale the parameter \( t \) so we may replace \( x_1 = \gamma t \) by \( x_1 = t \). \( \square \)

## 4 Tropical Polynomials

### 4.1 piecewise-linear functions with integer coefficients

Let \( x_1, x_2, \ldots, x_n \) be variables representing elements in \((\mathbb{R} \cup \{\infty\}, \oplus, \ominus)\), then a tropical monomial is a product of variables, allowing repetition. Consider for example:

\[
x_2 \odot x_1 \odot x_3 \odot x_1 \odot x_4 \odot x_2 \odot x_3 \odot x_2 = x_1^{\odot 2} x_2^{\odot 3} x_3^{\odot 2} x_4.
\]

As a function, a tropical monomial is a linear function.

\[
x_2 + x_1 + x_3 + x_1 + x_4 + x_2 + x_3 + x_2 = 2x_1 + 3x_2 + 2x_3 + x_4.
\]

Every linear function in \( n \) variables with integer coefficients we can write as a tropical monomial, exponents can be negative. Tropical monomials are the linear functions on \( \mathbb{R}^n \) with integer coefficients.

Any **finite** linear combination of tropical monomials defines a tropical polynomial with real coefficients and integer exponents:

\[
p(x_1, x_2, \ldots, x_n) = c_{a_1} \odot x_1^{\odot a_{1,1}} x_2^{\odot a_{1,2}} \cdots x_n^{\odot a_{1,n}} \\
+ c_{a_2} \odot x_1^{\odot a_{2,1}} x_2^{\odot a_{2,2}} \cdots x_n^{\odot a_{2,n}} \\
+ \cdots \\
+ c_{a_k} \odot x_1^{\odot a_{k,1}} x_2^{\odot a_{k,2}} \cdots x_n^{\odot a_{k,n}}.
\]

The corresponding function is

\[
p(x_1, x_2, \ldots, x_n) = \min\{ c_{a_1} + a_{1,1} x_1 + a_{1,2} x_2 + \cdots + a_{1,n} x_n, \\
  c_{a_2} + a_{2,1} x_1 + a_{2,2} x_2 + \cdots + a_{2,n} x_n, \\
  \cdots, \\
  c_{a_k} + a_{k,1} x_1 + a_{k,2} x_2 + \cdots + a_{k,n} x_n \}.
\]

Tropical polynomials as functions \( p : \mathbb{R}^n \to \mathbb{R} \) are continuous; are piecewise-linear, with a finite number of pieces; and are concave, i.e.: \( p \left( \frac{x + y}{2} \right) \geq \frac{1}{2} (p(x) + p(y)) \) for all \( x, y \in \mathbb{R} \). Any function which satisfies these three properties can be represented as the minimum of a finite set of linear functions.
Lemma 4.1 The tropical polynomials in $n$ variables are the piecewise-linear functions on $\mathbb{R}^n$ with integer coefficients.

Let us consider a tropical polynomial in one variable. In particular, we consider the graph of a tropical quadratic polynomial:

$$a \odot x^2 + b \odot x + c = \min(a + 2x, b + x, c)$$  \hspace{1cm} (18)

Observe the following:

If $b - a \leq c - b$, then $p(x) = a \odot (x \oplus (b - a)) \odot (x \oplus (c - b))$

$$= a + \min(x, b - a) + \min(x, c - b).$$  \hspace{1cm} (20)

Figure 5: The graph of $a \odot x^2 + b \odot x + c$.

Theorem 4.1 (tropical fundamental theorem of algebra) Every tropical polynomial function can be written uniquely as a tropical product of tropical linear functions.

The word function is key in the formulation of the theorem, as distinct polynomials can represent the same function:

$$x^2 \oplus 17 \odot x \oplus 2 = \min(2x, x + 17, 2) = \min(2x, x + 1, 2) = x^2 \oplus 1 \odot x \oplus 2 = (x \oplus 1)^2$$

Every polynomial can be replaced by an equivalent polynomial (equivalent means: representing the same function) that can be factored into linear functions.

4.2 graphing a tropical line

The graph of the tropical line $L(x, y) = 1 \odot x \oplus 2 \odot y \oplus 3$ is shown in Figure 6.

The tropical line $L(x, y) = 1 \odot x \oplus 2 \odot y \oplus 3$ defines the function:

$$L : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto \min(1 + x, 2 + y, 3).$$  \hspace{1cm} (21)
Looking where the minimum is attained at least twice yields a picture in the plane, see Figure 7.

Intersecting $1 \odot x \oplus 2 \odot y \oplus 3$ with $-2 \odot x \oplus 4 \odot y \oplus 2$: where $\min(1 + x, 2 + y, 3)$ and $\min(-2 + x, 4 + y, 2)$ attain their minimum twice is shown in Figure 8.
5 Exercises

1. Consider the amoeba of the product of two linear equations. How many tentacles do you see in each direction? For which choices of the coefficients do you see holes in the amoeba?

2. Prove the tropical fundamental theorem of algebra.

3. Investigate the complexity of graphing a tropical plane curve of degree $d$ and express the complexity as a function of $d$.

References


