Tropical Algebraic Geometry 2

This note is based on [1, 2], [4], and [5].

1 Unimodular Coordinate Transformations

Bergman’s theorem [4] answered a conjecture of Zalessky. Let $S = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ be the Laurent polynomial ring and $g = (g_{i,j}) \in \text{GL}(n, \mathbb{Z})$ invertible integer matrix defines the action

$$
    g : S \to S : x_i \mapsto \prod_{j=1}^{n} x_j^{g_{i,j}}
$$

$I$ is a proper ideal in $S$ and the stabilizer group of $I$ is

$$
    \text{Stab}(I) = \{ g \in \text{GL}(n, \mathbb{Z}) : gI = I \}.\tag{2}
$$

**Theorem 1.1 (Theorem 1 of Bergman 1971)** Let $I$ be a nontrivial ideal in $K[x^{\pm 1}]$, and $H \subseteq \text{GL}(n, \mathbb{Z})$ the stabilizer subgroup of $I$. Then $H$ has a subgroup $H_0$ of finite index, which stabilizes a nontrivial subgroup of $\mathbb{Z}^n$ (equivalently, which can be put into block-triangular form

$$
    \left( \begin{array}{cc}
        * & \ast \\
        0 & \ast
    \end{array} \right)
$$

in $\text{GL}(n, \mathbb{Z})$).

Bergman’s conceptual proof for $K = \mathbb{C}$ is summarized below.

Consider $V \subseteq (\mathbb{C} \setminus \{0\})^n$ defined by some nontrivial ideal.

1. Look at limiting values of ratios $\log |x_1| : \log |x_2| : \cdots : \log |x_n|$ as $x \in V$ becomes large. Identify this set of ratios with the $(n - 1)$-sphere $S^{n-1}$.

2. The limiting ratios of logarithms lies in a finite union of proper great subspheres on $S^{n-1}$, having rational defining parameters.

3. Assuming this, note: the intersection of two such finite unions of subspheres will again be one; and the family of all finite unions of great subspheres has a descending chain condition. There exists a unique finite union $U$ of subspheres minimal for the property of containing all “logarithmic limit-points at infinity” of $V$. If $V$ has positive dimension, $U$ must be nonempty.

4. The space of our $n$-tuples of logarithms $\mathbb{R}^n$ arises as the dual of $\mathbb{Z}^n$, that is: $\text{Hom}_{\text{groups}}(\mathbb{Z}^n, \mathbb{R})$. Thus we get a natural action of $\text{GL}(n, \mathbb{Z})$ on $\mathbb{R}^n$, and so on $S^{n-1}$.

5. Clearly $U$ will be invariant under the induced action of the stabilizer subgroup, $H$, of $I$.

By duality, we obtain from the great subspheres of $U$ a family $Q$ of nontrivial subgroups of $\mathbb{Z}^n$, also invariant under $H$.

The claim that logarithmic points at infinity of $V$ lie in a finite union of proper great subspheres of $S^{n-1}$, consider the support $A$ of any nonzero $f \in I$. At $z \in V$: $f(z) = \sum_{a \in A} c_a z^a = 0$. At each point of $V$, at least two terms of the sum (the largest ones) must be of the same order of magnitude. Each $\log |z|$ lies in one of the finite family of “planks” in $\mathbb{R}^n$.\[ \square \]
Unimodular coordinate transformations are defined by invertible integer matrices. The Lemma below appears in [5].

**Lemma 1.1** Denote the standard unit vectors by $e_1, e_2, \ldots$

1. Given any $v \in \mathbb{Z}^n$ with $\gcd(|v_1|, |v_2|, \ldots, |v_n|) = 1$. There is a matrix $U \in \text{GL}(n, \mathbb{Z})$: $Uv = e_1$.

2. Let $L$ be a rank $k$ subgroup of $\mathbb{Z}^n$ with $\mathbb{Z}^n/L$ torsion-free. There is a matrix $U \in \text{GL}(n, \mathbb{Z})$ with $UL$ equal to the subgroup generated by $e_1, e_2, \ldots, e_k$.

To prove the first statement:

\[
1 = \gcd(v_1, v_2) = av_1 + bv_2 = \begin{bmatrix} a & b \\ -v_1 & v_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]  

(3)

Apply $n - 1$ times repeatedly for a vector of length $n$.

Before continuing the proof, we investigate the notion of torsion-free. In a $\mathbb{Z}$-module, like a vector space we have scalar multiplication, but $\mathbb{Z}$ is a ring, not a field.

A group $G$ is torsion-free if $\forall g \in G \setminus \{0\}$ and $\forall n \in \mathbb{Z} \setminus \{0\}: ng \neq 0$. For $n \in \mathbb{Z} \setminus \{0\}$: $\mathbb{Z}/n\mathbb{Z}$ is not torsion-free.

Figure 1 shows an example of a lattice. We have:

\[
L = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \quad G = \mathbb{Z}^2/L = \langle g_1, g_2 \rangle/ \left( \begin{array}{l} 2g_1 - g_2 = 0 \\ g_1 + 2g_2 = 0 \end{array} \right) \sim \mathbb{Z}/5\mathbb{Z}
\]  

(4)

![Figure 1: An example of a lattice.](image)

Now we can continue the proof and show that: Let $L$ be a rank $k$ subgroup of $\mathbb{Z}^n$ with $\mathbb{Z}^n/L$ torsion-free. There is a matrix $U \in \text{GL}(n, \mathbb{Z})$ with $UL$ equal to the subgroup generated by $e_1, e_2, \ldots, e_k$.

Let $A \in \mathbb{Z}^{k \times n}$ contains in its rows a basis for $L$.

\[
\mathbb{Z}^n/L \text{ is torsion-free} \Rightarrow \text{Smith Normal Form (SNF) of } A = A' = [I \ 0],
\]  

(5)

where $I$ is the identity matrix. By SNF: $A' = V A U'$, for $V \in \text{GL}(k, \mathbb{Z})$ and $U' \in \text{GL}(n, \mathbb{Z})$. Because multiplication by invertible matrix does not change row span, the row span of $VA$ is the same as the row span of $L$.

\[
A' = [I \ 0] = [e_1 \ e_2 \ \cdots \ e_k]^T = (VA)U'
\]  

(6)

As $A'^T = U'^T(VA)^T$, take $U = U'^T$. \qed
2 the cyclic $n$-roots problem

The cyclic $n$-roots system is defined as

$$
f(x) = \begin{cases} 
    x_0 + x_1 + \cdots + x_{n-1} = 0 \\
    x_0 x_1 + x_1 x_2 + \cdots + x_{n-2} x_{n-1} + x_{n-1} x_0 = 0 \\
    \vdots \\
    x_0 x_1 x_2 \cdots x_{n-1} - 1 = 0.
\end{cases}
$$

Backelin showed in [3] the following:

**Lemma 2.1 (Backelin)** *If $m^2$ divides $n$, then the cyclic $n$-roots system has a solution set of dimension $m-1$.*

Methods to compute the positive dimensional components with polyhedral methods are outlined in [1] and [2].

An initial form of cyclic 9-roots is defined by the pair of vectors:

$$
v_0 = (1, 1, -2, 1, 1, -2, 1, 1, -2)
$$

$$
v_1 = (0, 1, -1, 0, 1, -1, 0, 1, -1),
$$

defining the initial form system

$$
\begin{align*}
x_2 + x_5 + x_8 &= 0 \\
x_0 x_1 x_2 + x_0 x_1 x_3 + x_0 x_1 x_4 &= 0 \\
x_0 x_1 x_2 x_3 + x_0 x_1 x_2 x_4 + x_0 x_1 x_2 x_5 + x_0 x_1 x_2 x_6 + x_0 x_1 x_2 x_7 + x_0 x_1 x_2 x_8 + x_0 x_1 x_2 x_9 &= 0 \\
x_0 x_1 x_2 x_3 x_4 + x_0 x_1 x_2 x_3 x_5 + x_0 x_1 x_2 x_3 x_6 + x_0 x_1 x_2 x_3 x_7 + x_0 x_1 x_2 x_3 x_8 + x_0 x_1 x_2 x_3 x_9 &= 0 \\
x_0 x_1 x_2 x_3 x_4 x_5 + x_0 x_1 x_2 x_3 x_4 x_6 + x_0 x_1 x_2 x_3 x_4 x_7 + x_0 x_1 x_2 x_3 x_4 x_8 + x_0 x_1 x_2 x_3 x_4 x_9 &= 0 \\
x_0 x_1 x_2 x_3 x_4 x_5 x_6 + x_0 x_1 x_2 x_3 x_4 x_5 x_7 + x_0 x_1 x_2 x_3 x_4 x_5 x_8 + x_0 x_1 x_2 x_3 x_4 x_5 x_9 &= 0 \\
x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7 + x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_8 + x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_9 &= 0 \\
x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 + x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_9 &= 0 \\
x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 &= 0
\end{align*}
$$

The unimodular transformation $x = y^M$ is

$$
M = \begin{bmatrix}
    1 & 1 & -2 & 1 & 1 & -2 & 1 & 1 & -2 \\
    0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 \\
    0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
    x_0 \\
    x_1 \\
    x_2 \\
    x_3 \\
    x_4 \\
    x_5 \\
    x_6 \\
    x_7 \\
    x_8
\end{bmatrix} = \begin{bmatrix}
    y_0 \\
    y_1 \\
    y_2 \\
    y_3 \\
    y_4 \\
    y_5 \\
    y_6 \\
    y_7 \\
    y_8
\end{bmatrix}
$$

We use the coordinate change to transform the initial form system and the original cyclic 9-roots system. The transformed initial form system is
A solution is $y_2 = -\frac{1}{2} - \frac{\sqrt{3}i}{2}$, $y_3 = -\frac{1}{2} + \frac{\sqrt{3}i}{2}$, $y_4 = 1$, $y_0 = -\frac{1}{2} - \frac{\sqrt{3}i}{2}$, $y_5 = -\frac{1}{2} + \frac{\sqrt{3}i}{2}$, $\sqrt{m}$, where $I = \sqrt{-1}$.

An exact representation of a two dimensional set of cyclic 9-roots is

$$
\begin{align*}
x_0 &= y_0 \\
x_1 &= y_1 \\
x_2 &= y_0^2 y_1^{-1} y_2 \\
x_3 &= y_0 y_3 \\
x_4 &= y_0 y_4 \\
x_5 &= y_0^2 y_1^{-1} y_5 \\
x_6 &= y_0 y_6 \\
x_7 &= y_0 y_7 \\
x_8 &= y_0^{-2} y_1^{-1} y_8
\end{align*}
$$

(12)

$$
\begin{align*}
x_0 &= t_1 \\
x_1 &= t_2 \\
x_2 &= t_1^{-2} t_2^{-1} (-\frac{1}{2} - \frac{\sqrt{3}i}{2}) \\
x_3 &= t_1(-\frac{1}{2} + \frac{\sqrt{3}i}{2}) \\
x_4 &= t_1 t_2 (-\frac{1}{2} + \frac{\sqrt{3}i}{2}) \\
x_5 &= t_1 t_2 (-\frac{1}{2} - \frac{\sqrt{3}i}{2}) \\
x_6 &= t_1(-\frac{1}{2} - \frac{\sqrt{3}i}{2}) \\
x_7 &= t_1 t_2 (-\frac{1}{2} - \frac{\sqrt{3}i}{2}) \\
x_8 &= t_1 t_2 (-\frac{1}{2} + \frac{\sqrt{3}i}{2})
\end{align*}
$$

(13)

Lemma 2.2 (Tropical Version of Backelin’s Lemma) For $n = m^2 \ell$, where $\ell \in \mathbb{N} \setminus \{0\}$ and $\ell$ is no multiple of $k^2$, for $k \geq 2$, there is an $(m-1)$-dimensional set of cyclic $n$-roots, represented exactly as

$$
\begin{align*}
&x_{km+0} = u^k t_0 \\
&x_{km+1} = u^k t_0 t_1 \\
&x_{km+2} = u^k t_0 t_1 t_2 \\
&\vdots \\
&x_{km+m-2} = u^k t_0 t_1 \ldots t_{m-2} \\
&x_{km+m-1} = \gamma u^k t_0^{m+1} t_1^{m+2} \ldots t_{m-3}^{m-4} \ldots t_{m-2}^{-2} t_{m-1}^{-1}
\end{align*}
$$

for $k = 0, 1, 2, \ldots, m-1$, free parameters $t_0, t_1, \ldots, t_{m-2}$, constants $u = e^{i2\pi \alpha}$, $\gamma = e^{i2\pi \beta}$, with $\beta = (\alpha \mod 2)$, and $\alpha = m(m\ell - 1)$. 
3 Polyhedral Geometry

A Newton polygon with inner normals to its edges is shown in Figure 2. The inner normal fan is shown at the left: the rays are normal to the edges of the polygon; and normals to the vertices of the polygon are contained in the strict interior of cones spanned by the rays.

Let \( P \) be an \( n \)-dimensional polytope. Denote the inner product by \( \langle \cdot, \cdot \rangle \). For \( \mathbf{v} \neq 0 \), the \textit{face of} \( P \) \textit{defined by} \( \mathbf{v} \) is

\[
\text{in}_\mathbf{v}(P) = \{ \mathbf{a} \in P \mid \langle \mathbf{a}, \mathbf{v} \rangle = \min_{\mathbf{b} \in P} \langle \mathbf{b}, \mathbf{v} \rangle \}. \tag{14}
\]

The \( \text{in}_\mathbf{v}(\cdot) \) notation refers to inner forms of polynomials that are supported on faces of the Newton polytopes. If we have a face \( F \) of \( P \), then its \textit{inner normal cone} is

\[
\text{cone}(F) = \{ \mathbf{v} \in \mathbb{R}^n \mid \text{in}_\mathbf{v}(P) = F \}. \tag{15}
\]

Passing from a face to its normal cone is like passing to the dual. Taking the dual of the dual brings us back to the original.

The \textit{Minkowski sum} of two sets \( A, B \subset \mathbb{R}^n \):

\[
A + B = \{ \mathbf{a} + \mathbf{b} \mid \mathbf{a} \in A, \mathbf{b} \in B \}. \tag{16}
\]

The Newton polytope of the product of two polynomials is the Minkowski sum of their Newton polytopes.

The \textit{common refinement} of two polyhedral fans \( \mathcal{F} \) and \( \mathcal{G} \) is

\[
\mathcal{F} \land \mathcal{G} = \{ P \cap Q \mid P \in \mathcal{F}, Q \in \mathcal{Q} \}. \tag{17}
\]

The normal fan of the Minkowski sum of two polytopes is the common refinement of their normal fans.

Let \( P = \text{conv}(\mathbf{a}_i, i = 1, 2, \ldots, r) \subset \mathbb{R}^n \). A \textit{regular subdivision} of \( P \) is induced by \( \mathbf{w} = (w_1, w_2, \ldots, w_r) \): \( \hat{P} = \text{conv}(\mathbf{a}_i, w_i) \mid i = 1, 2, \ldots, r \). Projecting the facets on the lower hull of \( \hat{P} \) onto \( \mathbb{R}^n \) — dropping the last coordinate — gives the cells in the regular subdivision induced by \( \mathbf{w} \). If all cells are simplices (spanned by exactly \( n + 1 \) points), then the regular subdivision is a regular triangulation.

A \textit{polyhedral complex} \( \mathcal{C} \) is a collection of polyhedra:

1. If a polyhedron \( P \in \mathcal{C} \), then for all \( \mathbf{v} \): \( \text{in}_\mathbf{v}(P) \in \mathcal{C} \).
2. If \( P, Q \in \mathcal{C} \), then either \( P \cap Q = \emptyset \) or \( P \cap Q \) is a face of both.

Polytopes, fans, and subdivisions are polyhedral complexes.
The computation of the convex hull is a major problem solved by computational geometry. Problem specification:

- input: a collection of points in the plane or in space;
- output: a description of all faces of the convex hull.

Solution: apply the beneath-beyond or the giftwrapping method. Software: Qhull.

In optimization, the linear programming method solves

\[
\min_{x} \langle c, x \rangle \quad \text{subject to} \quad Ax \geq b
\]  

(18)

Inner normals to facets are subject to a system of linear inequalities. Software: cddlib, lrs.

A nonzero vector \( v \) is a pretropism for the system \( f(x) = 0 \) if \( \#n(v) \geq 2 \) for all \( k = 0, 1, \ldots, N - 1 \).

Given a tuple of Newton polytopes \( P \) of a system \( f(x) = 0 \), the tropical prevariety of \( f \) is the common refinement of the normal cones to the edges of the Newton polytopes in \( P \).

A cone of pretropism is a polyhedral cone spanned by pretropisms.

If we are looking for an algebraic set of dimension \( d \) and

- if there are no cones of vectors perpendicular to edges of the Newton polytopes of \( f(x) = 0 \) of dimension \( d \), then the system \( f(x) = 0 \) has no solution set of dimension \( d \) that intersects the first \( d \) coordinate planes properly; otherwise
- if a \( d \)-dimensional cone of vectors perpendicular to edges of the Newton polytopes exists, then that cone defines a part of the tropical prevariety.

For the cyclic 9-roots system, we found a two dimensional cone of pretropisms.

References


