Midterm Exam MCS 563 Jan Verschelde Wednesday 2 March 2011

This is a take home exam. Your answers to the questions below are due on Friday 4 March, at 10AM (bring your answers to class). Every question counts for 20 points.

1. Consider the following polynomial system:

$$f(x,y,z) = \begin{cases} y - x^2 = 0 \\ z - x^3 = 0 \\ x^2 + y^2 + z^2 - r^2 = 0 \end{cases} \quad r \in \mathbb{R}, r \geq 0.$$

   (a) For any positive value of the parameter r, how many solutions do you expect by the plain application of the theorem of Bézout? Does a multihomogeneous version of Bézout's theorem give a sharper bound? Compare the bound to the actual number of solutions.

   (b) Use resultants to express the solutions of $f(x,y,z) = 0$ in terms of the parameter r. Does the order of variables in the elimination matter?

2. Let $V = \{(1,2,0),(2,3,1),(0,1,2),(2,1,-1),(3,2,3)\}$.

   (a) Give a shape lemma representation for the ideal $I(V)$.

   (b) Using interpolation, reduce $f = x_2^2 x_3 - 3x_1 x_3^2 + 2x_1^2$ modulo I, w.r.t. $<_{\rm lex}$.

3. Consider $h(x,y,t) = \begin{cases} xy - y - 1 = 0 \\ (1-t)(x+1) + t(x-1) = 0 \end{cases}$ for t going from 0 to 1.

   (a) At $t = 1/2$, consider the solution $(x,y) = (0,-1)$. Compute the tangent vector to the path at $(0,-1)$ and the predicted solution for step length equal to 1/10.

   (b) Do the same as in (a) at $t = 99/100$. Compare Δt with the Δt of (a) for the same step length 1/10 and explain the difference.

4. Consider $f = x^4 - x^3 y + 3x^2 y^3 - 3xy^5 + y^7 \in \mathbb{C}[[x]][y]$.

   (a) Draw the Newton polyhedron of f and find the leading powers for the Puiseux series expansions.

   (b) For one of the leading powers you obtained in (a), compute the first and second coefficient of the Puiseux series.

5. Given is the polynomial system

$$f(x_1,x_2) = \begin{cases} c_{32} x_1^3 x_2^2 + c_{10} x_1 + c_{01} x_2 = 0 \\ c_{23} x_1^2 x_2^3 + c_{02} x_2^2 + c_{00} = 0 \end{cases}$$

   where the six coefficients c_{32}, c_{10}, c_{01}, c_{23}, c_{02}, and c_{00} are nonzero complex numbers.

   (a) Use the Newton polygons of the polynomials in f to compute an upper bound for the number of isolated solutions of the system $f(x_1,x_2) = \mathbf{0}$.

   (b) Is this bound sharp for all nonzero choices of the six coefficients? Justify your answer.