Midterm Exam MCS 563 Jan Verschelde Monday 2 March 2009

This is a take home exam. Your answers to the questions below are due on Wednesday 4 March, at 11AM (bring your answers to class). Every question counts for 20 points.

1. Consider the system

$$f(x_1, x_2, x_3) = \begin{cases} x_1^3 x_2 + 2x_1^2 x_3 + 3x_1 x_2 x_3 & = & 0 \\ x_2^3 x_3 + 2x_2^2 x_1 + 3x_2 x_3 x_1 & = & 0 \\ x_3^3 x_1 + 2x_3^2 x_2 + 3x_3 x_1 x_2 & = & 0. \end{cases} \quad (1)$$

   (a) Describe the symmetric structure of this system.
   Apply Bézout's theorem to bound the number of solutions.

   (b) Set up a symmetric linear-product start system and explain the orbit structure of the solutions in a homotopy that respects the symmetry.

2. Consider the intersection of a circle with a parabola.

   (a) Write equations for the intersection problem, using parameters for the circle and parabola. Compute the equation for the discriminant of this problem.

   (b) Sample a couple of points on the discriminant variety and given an interpretation of the singular instances of the intersection problem.

3. Consider the system

$$f(x_1, x_2) = \begin{cases} 1.5365\ 10^{56} x_1^3 x_2 + 1.3611\ 10^{29} x_1^2 - 1.1130\ 10^{14} x_2 & = & 0 \\ 1.8237\ 10^{68} x_1^2 x_2^3 - 2.2753\ 10^{27} x_2^2 + 12 & = & 0. \end{cases} \quad (2)$$

   (a) Apply variable scaling to reduce the variability among the coefficients.

   (b) Explain and illustrate how this scaling improves the numerical conditioning of the solutions.

4. Let $V = \{(1,4), (2,3), (-3,0), (5,1)\}$ be a solution set.

   (a) Give the shape lemma representation for $I(V)$.

   (b) Using interpolation, reduce $f = x_1^4 x_2^3 - 7x_1^3 x_2^5 + 3x_1^2$ modulo I, w.r.t. $<_{\rm lex}$.

5. Consider the system

$$f(x_1, x_2) = \begin{cases} x_1^2 x_2^2 - x_1^2 x_2 + 6 & = & 0 \\ -x_1^3 x_2 + x_1^3 - 13 & = & 0. \end{cases} \quad (3)$$

   (a) Apply Bernshteĭn's theorem to bound the number of isolated solutions.

   (b) Show that the system has at least one solution at infinity.