Multihomogeneous Homotopies

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   - number of equilibria in a non-cooperative game
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   - row expansion for permanents
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MCS 563 Lecture 7
Analytic Symbolic Computation
Jan Verschelde, 26 January 2011

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the eigenvalue problem

Consider a 2-by-2 eigenvalue problem $\lambda \mathbf{x} = A\mathbf{x}$ viewed as a polynomial system:

$$f(\lambda, \mathbf{x}) = \left\{ \begin{array}{r} \lambda x_1 - a_{11} x_1 - a_{12} x_2 = 0 \\ \lambda x_2 - a_{21} x_1 - a_{22} x_2 = 0 \\ c_0 + c_1 x_1 + c_2 x_2 = 0. \end{array} \right.$$

For general $A \in \mathbb{C}^{2 \times 2}$ and random c_0, c_1, and c_2, the system has two isolated solutions.

For any dimension n: $\lambda \mathbf{x} = A\mathbf{x}$ as n quadratic equations

- has Bézout bound 2^n, but
- only n eigenvalue-eigenvector pairs.

Embedding $\lambda \mathbf{x} = A\mathbf{x}$ in $\mathbb{P} \times \mathbb{P}^n$ yields a sharp bound.

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multihomogeneous coordinates

The embedding separates the eigenvalues from the coordinates of the eigenvector.
Replacing λ by λ_1/λ_0 and x_1 by z_1/z_0, x_2 by z_2/z_0. and clearing denominators:

$$f(([\lambda_0 : \lambda_1], [z_0 : z_1 : z_2])) = \left\{ \begin{array}{r} \lambda_1 z_1 - a_{11}\lambda_0 z_1 - a_{12}\lambda_0 z_2 = 0 \\ \lambda_1 z_2 - a_{21}\lambda_0 z_1 - a_{22}\lambda_0 z_2 = 0 \\ c_0 z_0 + c_1 z_1 + c_2 z_2 = 0. \end{array} \right.$$

$([\lambda_0 : \lambda_1], [z_0 : z_1 : z_2]) \in \mathbb{P}^1 \times \mathbb{P}^2$ are equivalence classes: we can scale the coordinates of the eigenvalues independently of the coordinates of the eigenvector.

As with $\mathbb{P}^n$, at least one coordinate must be nonzero.

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Solutions at infinity are solutions with $\lambda_0 = 0$ or $z_0 = 0$.

1. If $\lambda_0 = 0$, then as $\lambda_1 \neq 0$ we must have that $z_1 = 0$ and $z_2 = 0$, but then also $z_0 = 0$, which cannot be.
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Multihomogeneous Homotopies

a non-cooperative game

Take a non-cooperative game with given

1. three players: a, b, and c;
2. two strategies for each player;
3. payoffs for players a, b, c are $A, B, C \in \mathbb{R}^{2 \times 2 \times 2}$.
   A_{ijk} is the payoff to a if a, b, c chooses i, j, k.

Variable x_i is the probability player a chooses strategy i.
As probabilities $x_i \in [0,1]$ and $x_1 + x_2 = 1$.
Variables y_1, y_2 and z_1, z_2 are allocations for players b and c, also with $y_1 + y_2 = 1$ and $z_1 + z_2 = 1$.

Setting could be continuous, e.g.: how much percentage of money to allocate to an investment?
or discrete, e.g.: how many lectures to attend?

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computing the payoffs

For payoff matrix $A \in \mathbb{R}^{2 \times 2 \times 2}$, the payoff α for player a is

$$\alpha = \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} A_{ijk} x_i y_j z_k .$$

A vector $(x_1, x_2, y_1, y_2, z_1, z_2)$ is a Nash equilibrium if

- no player gets higher payoff by changing strategy
- while the other two players keep their strategies fixed.

Using $\alpha = x_1 \sum_{j=1}^{2} \sum_{k=1}^{2} A_{1jk} y_j z_k + x_2 \sum_{j=1}^{2} \sum_{k=1}^{2} A_{2jk} y_j z_k$, $x_1 + x_2 = 1$ and $x_1 \geq 0$, $x_2 \geq 0$, we obtain

$$x_1 \left(\alpha - \sum_{j=1}^{2} \sum_{k=1}^{2} A_{1jk} y_j z_k \right) = x_2 \left(\alpha - \sum_{j=1}^{2} \sum_{k=1}^{2} A_{2jk} y_j z_k \right) = 0.$$

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a polynomial system

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is equivalent to

$$\sum_{j=1}^{2} \sum_{k=1}^{2} A_{1jk} y_j z_k - \sum_{j=1}^{2} \sum_{k=1}^{2} A_{2jk} y_j z_k = 0.$$

Similarly, we have equations using payoffs B and C.
Eliminating x_2, y_2, z_2 with $1 - x_1, 1 - y_1, 1 - z_1$,
we obtain 3 quadratic *bilinear* equations in 3 unknowns.

Embedding (x_1, y_1, z_1) in $\mathbb{P} \times \mathbb{P} \times \mathbb{P}$ (3-homogenization) gives a generically sharp Bézout bound.

Multihomogeneous Homotopies

multihomogeneous structures

We partition $\{\lambda, x_1, x_2\}$ into $\{\{\lambda\}, \{x_1, x_2\}\}$:

$$\left\{ \begin{array}{lcr} f_1 & : & \lambda x_1 - a_{11}x_1 - a_{12}x_2 = 0 \\ f_2 & : & \lambda x_2 - a_{21}x_1 - a_{22}x_2 = 0 \\ f_3 & : & c_0 + c_1x_1 + c_2x_2 = 0 \end{array} \right.$$

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for random coefficients α_{ij} and β_{ij}, which leads to

$$g(\lambda, x_1, x_2) = \left\{ \begin{array}{r} (\alpha_{10} + \alpha_{11}\lambda_1)(\beta_{10} + \beta_{11}x_1 + b_{12}x_2) = 0 \\ (\alpha_{20} + \alpha_{21}\lambda_2)(\beta_{20} + \beta_{21}x_1 + \beta_{22}x_2) = 0 \\ \beta_{30} + \beta_{31}x_1 + \beta_{32}x_2 \ = 0. \end{array} \right.$$

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linear-product start systems

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we make the same moves as a formal root count:

$$\begin{aligned} B_{\{\{\lambda\},\{\mathbf{x}\}\}} &= 1_{(g_1,\lambda)} \times 1_{(g_2,\mathbf{x})} \times 1_{(g_3,\mathbf{x})} \\ &\quad + 1_{(g_2,\lambda)} \times 1_{(g_1,\mathbf{x})} \times 1_{(g_3,\mathbf{x})} \\ &= 2. \end{aligned}$$

The subscripts indicate a factor in $g(\lambda, \mathbf{x}) = \mathbf{0}$.

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a multihomogeneous Bézout number

For any partition $Z = \{Z_1, Z_2, \ldots, Z_m\}$ of the set of unknowns, we compute the degrees $\deg(f_i, Z_j)$ of $f(\mathbf{x}) = \mathbf{0}$.

A linear-product start system that respects the multihomogeneous structure of f has equations:

$$g_i(\mathbf{x}) = \prod_{j=1}^{m} \prod_{k=1}^{\deg(f_i,Z_j)} \left(c_{ijk0} + \sum_{x_\ell \in Z_j} c_{ijk\ell} x_\ell \right) = 0$$

for random complex coefficients c_{ijk0} and $c_{ijk\ell}$.

The number of solutions of $g(\mathbf{x}) = \mathbf{0}$ equals B_Z, the m-homogeneous Bézout number of f, with respect to the partition Z.

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Multihomogenous Bézout's Theorem

Theorem (multihomogenous theorem of Bézout)

Let Z be any partition of the set of unknowns $\mathbf{x}$ of the system $f(\mathbf{x}) = \mathbf{0}$ and let B_Z be its Bézout bound. Then the number of isolated solutions of $f(\mathbf{x}) = \mathbf{0}$ in $\mathbb{C}^n$ is bounded by B_Z.

Follows from the correctness of a homotopy method:

- Let $g(\mathbf{x}) = \mathbf{0}$ be a linear-product start system for Z.
- All solutions of $f(\mathbf{x}) = \mathbf{0}$ lie at the end of paths defined by the homotopy

$$h(\mathbf{x}, t) = \gamma(1 - t)g(\mathbf{x}) + t\, f(\mathbf{x}) = \mathbf{0},$$

where $\gamma \in \mathbb{C}$ is a random complex constant.

Multihomogenous Bézout's Theorem

Theorem (multihomogenous theorem of Bézout)

Let Z be any partition of the set of unknowns $\mathbf{x}$ of the system $f(\mathbf{x}) = \mathbf{0}$ and let B_Z be its Bézout bound. Then the number of isolated solutions of $f(\mathbf{x}) = \mathbf{0}$ in $\mathbb{C}^n$ is bounded by B_Z.

Follows from the correctness of a homotopy method:

- Let $g(\mathbf{x}) = \mathbf{0}$ be a linear-product start system for Z.
- All solutions of $f(\mathbf{x}) = \mathbf{0}$ lie at the end of paths defined by the homotopy

$$h(\mathbf{x}, t) = \gamma(1 - t)g(\mathbf{x}) + t\, f(\mathbf{x}) = \mathbf{0},$$

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Multihomogeneous Homotopies

1. Multiprojective Space
   - the eigenvalue problem
2. Nash Equilibria
   - number of equilibria in a non-cooperative game
3. Linear-Product Systems
   - multihomogeneous structures
   - a multihomogeneous Bézout number
4. Algorithms and Extensions
   - row expansion for permanents
   - general linear-product start systems

row expansion for permanents

Input: $A \in \mathbb{N}^{n \times m}$, $\mathbf{k} = (k_1, k_2, \ldots, k_m)$, $k_1 + k_2 + \cdots + k_m = n$;
i controls level of recursion, call with $i = 1$.
Output: $\mathrm{per}(A, \mathbf{k})$ permanent of A with respect to $\mathbf{k}$.

```
$s := 0$; $\mathbf{k}' := \mathbf{k}$;
for $j$ from 1 to $m$ do
   if $k'_j \neq 0$ and $a_{ij} \neq 0$ then
      $k'_j := k'_j - 1$;
      if $i = n$
       then $s := s + a_{ij}$;
       else $s := s + a_{ij} \times \mathrm{per}(A, \mathbf{k}', i+1)$;
      end if;
      $k'_j := k'_j + 1$;
   end if;
end for;
return $s$.
```

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end for;
return $s$.
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Multihomogeneous Homotopies

supporting set structures

For example, consider the following polynomial system:

$$f(\mathbf{x}) = \left\{ \begin{array}{l} x_1^2 x_2 + x_2^2 + x_1 + 1 = 0 \\ x_2^2 x_1 + x_1^2 + x_2 + 1 = 0 \end{array} \right. \qquad S = \left(\begin{array}{c} \{\{x_1\}, \{x_1, x_2\}, \{x_2\}\} \\ \{\{x_2\}, \{x_2, x_1\}, \{x_1\}\} \end{array} \right)$$

where S is a supporting set structure for f, which leads to

$$g(\mathbf{x}) = \left\{ \begin{array}{l} (c_{11}x_1 + c_{12})(c_{13}x_1 + c_{14}x_2 + c_{15})(c_{16}x_2 + c_{17}) = 0 \\ (c_{21}x_2 + c_{22})(c_{23}x_2 + c_{24}x_1 + c_{25})(c_{26}x_1 + c_{27}) = 0 \end{array} \right.$$

for random coefficients c_{ij}.

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a generalized Bézout number

Solving

$$g(\mathbf{x}) = \left\{ \begin{array}{l} (c_{11}x_1 + c_{12})(c_{13}x_1 + c_{14}x_2 + c_{15})(c_{16}x_2 + c_{17}) = 0 \\ (c_{21}x_2 + c_{22})(c_{23}x_2 + c_{24}x_1 + c_{25})(c_{26}x_1 + c_{27}) = 0 \end{array} \right.$$

can done as a formal calculation of a Bézout number B_S, based directly on the supporting set structure S:

$$\begin{array}{rcccccc} B_S & = & \underset{\{x_1\}\{x_2\}}{1} & + & \underset{\{x_1\}\{x_2,x_1\}}{1} & + & \underset{\{x_1,x_2\}\{x_2\}}{1} \\ & + & \underset{\{x_1,x_2\}\{x_2,x_1\}}{1} & + & \underset{\{x_1,x_2\}\{x_1\}}{1} & + & \underset{\{x_2\}\{x_2,x_1\}}{1} \\ & + & \underset{\{x_2\}\{x_1\}}{1} & = & 7. & & \end{array}$$

The sets underneath the formula indicate the sets associated with the linear systems in $g(\mathbf{x}) = \mathbf{0}$.

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permutation symmetry

Observe the symmetry in:

$$f(\mathbf{x}) = \left\{ \begin{array}{l} x_1^2 x_2 + x_2^2 + x_1 + 1 = 0 \\ x_2^2 x_1 + x_1^2 + x_2 + 1 = 0. \end{array} \right.$$

We have:

$$\left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \left[\begin{array}{c} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{array} \right] = f\left(\left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right] \right).$$

With a particular choice of the coefficients of

$$g(\mathbf{x}) = \left\{ \begin{array}{l} (c_1 x_1 + c_2)(c_3 x_1 + c_4 x_2 + c_5)(c_6 x_2 + c_7) = 0 \\ (c_1 x_2 + c_2)(c_3 x_2 + c_4 x_1 + c_5)(c_6 x_1 + c_7) = 0. \end{array} \right.$$

The start system has the same symmetry as $f(\mathbf{x}) = \mathbf{0}$.

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A Generalized Bézout Theorem

Regularity and boundedness of a homotopy starting at a linear-product start system proves:

Theorem

Let S be a supporting set structure for the system $f(\mathbf{x}) = \mathbf{0}$*. Let* B_S *be the number of solutions of a linear-product system* $g(\mathbf{x}) = \mathbf{0}$ *supported by S. Then the number of solutions in* $\mathbb{C}^n$ *of* $f(\mathbf{x}) = \mathbf{0}$ *is bounded by* B_S*.*

Geometric version of Bézout's theorem: degenerate hypersurfaces $f_i^{-1}(\mathbf{0})$ into a union of hyperplanes.

General linear-product start systems are advantageous to exploit permutation symmetry

$\Rightarrow$ it suffices to compute only the generating solutions.

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$\Rightarrow$ it suffices to compute only the generating solutions.

Summary + Exercises

We generalized the theorem of Bézout to use linear-product start systems and multihomogeneous structures.

Exercises:

1. Embed the system of the 2-dimensional eigenvalue problem in ordinary projective space and compute the two solutions at infinity.
2. For the 2-homogeneous 2-dimensional eigenvalue problem there are no solutions at infinity. Show that the same holds for a 2-homogeneous 3-dimensional eigenvalue problem. Can you generalize the arguments to an *n*-dimensional eigenvalue problem?

more exercises

3. The generalized eigenvalue problem considers $\beta A\mathbf{x} = \alpha B\mathbf{x}$, for matrices A, B and corresponding pairs (α, β) of generalized eigenvalues. Given two matrices A and B, we look for all pairs (α, β) for which a nonzero $\mathbf{x}$ satisfies $\beta A\mathbf{x} = \alpha B\mathbf{x}$. Compute a multihomogeneous Bézout bound for this problem. Is this bound sharp?
4. Write a MATLAB (or Octave) function to compute the permanent of a matrix. You may also of course use Maple or Sage. The input to this function consists of an n-by-m matrix (with $m \leq n$) and a vector $\mathbf{k}$ of m positive integers $\mathbf{k} = (k_1, k_2, \ldots, k_m)$, with $k_1 + k_2 + \cdots + k_m = n$. The number k_i is the cardinality of the ith set of variables.

and more exercises

⑤ Use a computer algebra system to generate a polynomial system to compute all Nash equilibria for three players with each two strategies. Use the following data for the payoff matrices:

	111	112	121	122	211	212	221	222
$A =$	0	6	11	1	6	4	6	8
$B =$	12	7	6	8	1	12	8	1
$C =$	11	11	3	3	0	14	2	7

Compute the 3-homogeneous Bézout bound and solve the system, using `phc`, available for download at `http://www.math.uic.edu/~jan/download.html`.