the Newton-Puiseux Method

1. Formal Power Series
   - normal forms
   - substitution
2. Parametrization of Algebraic Curves
   - series in an auxiliary variable t
   - reducibility and normal form
3. Fractional Power Series
   - the theorem of Puiseux
   - the Newton polygon

MCS 563 Lecture 15
Analytic Symbolic Computation
Jan Verschelde, 16 February 2011

the Newton-Puiseux Method

formal power series

Consider a domain D (e.g.: $\mathbb{Z}$) and polynomials $p(x) \in D[x]$.

Polynomials are finite sums, formal power series are infinite sums, denote the set of formal power series as $D[[x]]$.

Lemma 1

The series $a_0 + a_1 x + a_2 x^2 + \cdots$ has an inverse in $D[[x]]$ $\Leftrightarrow$ a_0 has an inverse in D.

In a field, every formal power series has a normal form:

Theorem 1

Let K be any field. Every element in $K[[x]]$ is of the form

$$x^{-h}(a_0 + a_1 x + a_2 x^2 + \cdots), \quad h \geq 0.$$

proof of Lemma 1

Proof. We compute the inverse of $a_0 + a_1 x + a_2 x^2 + \cdots$ denoting it as $b_0 + b_1 x + b_2 x^2 + \cdots$.
The coefficients of the inverse must satisfy

$$(a_0 + a_1 x + a_2 x^2 + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots) = 1.$$

Expanding the product above, we solve linear equations in the coefficients of the inverse:

$$\begin{aligned}
a_0 b_0 = 1 \quad &\Rightarrow \quad b_0 = a_0^{-1}, \text{ because } a_0^{-1} \text{ exists} \\
a_0 b_1 + a_1 b_0 = 0 \quad &\Rightarrow \quad b_1 = -a_1 b_0 a_0^{-1} \\
a_0 b_2 + a_1 b_1 + a_2 b_0 = 0 \quad &\Rightarrow \quad b_2 = -(a_1 b_1 + a_2 b_0) a_0^{-1}, etc.
\end{aligned}$$

The computation shows $\Rightarrow$. For $\Leftarrow$ to hold, we take $b_0 + b_1 x + b_2 x^2 + \cdots$ as the inverse of $a_0 + a_1 x + a_2 x^2 + \cdots$ and the product of the two leads to $a_0 b_0 = 1$, so b_0 is the inverse of a_0. □

normal forms

Theorem 1

Let K be any field. Every element in $K[[x]]$ is of the form

$$x^{-h}(a_0 + a_1 x + a_2 x^2 + \cdots), \quad h \geq 0.$$

Proof. Any $f \in K[[x]]$: $f = \dfrac{b_0 + b_1 x + b_2 x^2 + \cdots}{c_0 + c_1 x + c_2 x^2 + \cdots}$.

Let c_h be the smallest nonzero coefficient. As K is a field, c_h^{-1} exists and by Lemma 1 $c_h + c_{h+1} x + c_{h+2} x^2 + \cdots$ has an inverse $d_0 + d_1 x + d_2 x^2 + \cdots$, so we simplify f as

$$\begin{aligned} f &= \frac{(b_0 + b_1 x + b_2 x^2 + \cdots)(d_0 + d_1 x + d_2 x^2 + \cdots)}{x^h (c_h + c_{h+1} x + c_{h+2} x^2 + \cdots)(d_0 + d_1 x + d_2 x^2 + \cdots)} \\ &= \frac{a_0 + a_1 x + a_2 x^2 + \cdots}{x^h}. \end{aligned}$$

$\square$

order of a series

Theorem 1 justifies the definition of the order of a series.

Definition 1

For $f \in K[[x]]$, $f(x) = x^k(a_0 + O(x))$ or
$f(x) = a_0 x^k(1 + O(x))$, k is ***the order of f***, denoted by $O(f)$.

For $f, g \in K[[x]]$, we have $O(fg) = O(f) + O(g)$ and
$O(f \pm g) \geq \min(O(f), O(g))$.
For convenience, we take $O(o) = \infty$.

Another application of the Lemma 1 concerns factorization:

$$f, g \in K[[x]] : f | g \Leftrightarrow O(f) \leq O(g).$$

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substitution

Definition 2

$f, g \in K[[x]]$ ***are congruent modulo*** x^m, denoted by $f \equiv g (\mathrm{mod}\, x^m)$, if $f - g$ is divisible by x^m, or equivalently $O(f - g) \geq m$, or the first m coefficients of f and g are equal.

Theorem 2

1. *The congruence modulo* x^m *is an equivalence relation.*
2. *If* $f_1 \equiv f_2 (\mathrm{mod}\, x^m)$ *and* $g_1 \equiv g_2 (\mathrm{mod}\, x^m)$, *then* $f_1 \pm g_1 \equiv f_2 \pm g_2 (\mathrm{mod}\, x^m)$ *and* $f_1 g_1 \equiv f_2 g_2 (\mathrm{mod}\, x^m)$.
3. *If* $f \equiv g (\mathrm{mod}\, x^m)$ *for arbitrarily large m, then* $f = g$.
4. *If* f_1 *and* f_2 *are polynomials, if* $O(g_1) > 0$, $O(g_2) > 0$, *and if* $f_1 \equiv f_2 (\mathrm{mod}\, x^m)$, $g_1 \equiv g_2 (\mathrm{mod}\, x^m)$, *then* $f_1(g_1) \equiv f_2(g_2) (\mathrm{mod}\, x^m)$.
5. *If* $f_1, f_2, \ldots \in K[[x]]$: $f_{m+1} \equiv f_m (\mathrm{mod}\, x^m)$, $m = 1, 2, \ldots$, *then there exists a unique* $f \in K[[x]]$: $f_m \equiv f (\mathrm{mod}\, x^m)$, $m = 1, 2, \ldots$.

substitution in polynomials

4. Note that $O(g_1) > 0, O(g_2) > 0 \Rightarrow g_1(0) = 0 = g_2(0)$.

If f_1, f_2, g_1 and g_2 all agree for the first m coefficients, then we have to prove that the results of substituting g_1 into f_1 and g_2 into f_2 agree for the first m coefficients.

Since $f_1 \equiv f_2 (\text{mod } x^m)$, we can write $f_1 = f_2 + x^m f_3$, where f_3 makes up for all the different coefficients.

By #2, adding and multiplying congruent series produces congruent series, applied to $f_1(g_1)$ leads to $f_1(g_1) \equiv f_2(g_2) + g_2^m f_3(g_2) (\text{mod } x^m)$.

Then we compute the order

$$O(g_2^m f_3(g_2)) = m \underbrace{O(g_2)}_{\geq 1} + \underbrace{O(f_3(g_2))}_{\geq 0} \geq m.$$

Thus $f_1(g_1) \equiv f_2(g_2) (\text{mod } x^m)$.

term after term substitution

5. $f_1, f_2, f_3, f_4, \ldots$ have increasing similarities:
$f_2 \equiv f_1 (\mathrm{mod}\ x)$, $f_3 \equiv f_2 (\mathrm{mod}\ x^2)$, $f_4 \equiv f_3 (\mathrm{mod}\ x^3)$, $\ldots$
Denote the coefficient with x^j in f_i as a_{ij}, then:

$$\begin{aligned} f_1 &= a_{10} + a_{11}x + a_{12}x^2 + a_{13}x^3 + \cdots \\ f_2 &= a_{10} + a_{21}x + a_{22}x^2 + a_{23}x^3 + \cdots \\ f_3 &= a_{10} + a_{21}x + a_{32}x^2 + a_{33}x^3 + \cdots \\ f_4 &= a_{10} + a_{21}x + a_{32}x^2 + a_{43}x^3 + \cdots \end{aligned}$$

Selecting along the diagonal leads to

$$f = a_{10} + a_{21}x + a_{32}x^2 + a_{43}x^3 + \cdots + a_{m+1m}x^m + \cdots,$$

so we know f exists, we still have to prove f is unique.
Suppose $g \in K[[x]]$: $f_m \equiv g (\mathrm{mod}\ x^m)$. As $\equiv$ is an equivalence relation: $f \equiv g (\mathrm{mod}\ x^m)$. Since m can be arbitrarily large, we apply *#3* to arrive at $f = g$. □

substitution order

Theorem 3

1. *For fixed g, $f \rightarrow f(g)$ is a homomorphism of $K[[x]]$ into itself.*
2. *If $f(g) \neq 0$, $O(f(g)) = O(f)O(g)$.*
3. *If $O(g) > 0$, $O(h) > 0$: substituting h in $f(g)$ is the same as substituting $g(h)$ in f.*

Proof. #1 and #2 follow from the definition of substitution.

To prove #3, let $k = f(g)$ and $\ell = g(h)$, let f_m, g_m, h_m by polynomials congruent modulo x^m to the corresponding series f, g, and h; note that $k_m = f_m(g_m)$ and $\ell_m = g_m(h_m)$ are polynomials.

proof continued

As we want to substitute h in $f(g)$, consider $k_m(h_m)$ and as we want to substitute $g(h)$ in f, consider $f_m(\ell_m)$.

All of these are polynomials, so $k_m(h_m) = f_m(\ell_m)$.

We have: $k_m \equiv k (\mathrm{mod}\ x^m)$, $h_m \equiv h (\mathrm{mod}\ x^m)$, $f_m \equiv f (\mathrm{mod}\ x^m)$, $\ell_m \equiv \ell (\mathrm{mod}\ x^m)$, and also: $k_m(h_m) \equiv k(h) (\mathrm{mod}\ x^m)$, $f_m(\ell_m) \equiv f(\ell) (\mathrm{mod}\ x^m)$.

Then by equivalence it follows: $k(h) \equiv f(\ell) (\mathrm{mod}\ x^m)$ for arbitrary m, so finally: $k(h) = f(\ell)$. □

substition with $O(g) = 1$

Theorem 4

If $O(g_1) = 1$ and $f_2 = f_1(g_1)$, then

1. $O(f_2) = O(f_1)$.
2. *There is a g_2, $O(g_2) = 1$, such that $f_1 = f_2(g_2)$, for every $f_1 \in K[[x]]$.*

Proof. To show #*1*, apply #*2* of Theorem 3.

Let $g_1 = b_1 x + b_2 x^2 + \cdots$, $b_1 \neq 0$ and $g_2 = c_1 x + c_2 x^2 + \cdots$ with coefficients to be determined so that $g_1(g_2) = x$, then $f_2 = f_1(g_1)$, apply #*3* of Theorem 3: $f_2(g_2) = f_1(g_1(g_2)) = f_1$.

Now the proof is reduced to finding the coefficients c_i of g_2.

proof continued

The condition $g_1(g_2) = x$ leads to a recurrence on c_i:

$$
\begin{aligned}
g_1(g_2) &= b_1 g_2 + b_2 g_2^2 + b_3 g_2^3 + \cdots \\
&= b_1(c_1 x + c_2 x^2 + c_3 x^3 \cdots) \\
&\quad + b_2(c_1 x + c_2 x^2 + c_3 x^3 \cdots)^2 \\
&\quad + b_3(c_1 x + c_2 x^2 + c_3 x^3 \cdots)^3 + \cdots \\
&= b_1 c_1 x + (b_1 c_2 + b_2 c_1^2) x^2 \\
&\quad + (b_1 c_3 + 2 b_2 c_1 c_2 + b_3 c_1^3) x^3 + \cdots \\
&\quad + (b_1 c_n + P_n(b_2, \ldots, b_n, c_1, \ldots, c_{n-1})) x^n + \cdots
\end{aligned}
$$

where P_n is a polynomial, and for the coefficient with x^n to be zero: $c_n = -b_1^{-1} P_n(b_2, \ldots, b_n, c_1, \ldots, c_{n-1})$. □

the Newton-Puiseux Method

Parametrizations

Definition 3

Let $F(\mathbf{x}) = 0$ be the equation of an algebraic curve C in the projective plane. Then $x_0, x_1, x_2 \in K[[t]]$ are the coordinates of ***a parametrization of C*** if

1. $F([x_0 : x_1 : x_2]) = 0$; and
2. there is no $e \in K[[t]]$ such that $ex_i \in K$, for $i = 1, 2, 3$.

Definition 4

For any parametrization x_1, x_2, x_3 of C, let $h = -\min(O(x_i))$ and if $y_i = t^h x_i \in K[[t]]$ is the same parametrization, then $y_i(0) = a_i$ exists and at least one $a_i \neq 0$. The point $[a_0 : a_1 : a_2]$ is ***the center of the parametrization.***

equivalence classes

Definition 5

Let x_0, x_1, x_2 be a parameterization and $s \in K[[t]]$, $s \neq 0$, and $O(s) > 0$, then $y_i = x_i(t)$ is a parametrization with the same center. If $O(s) = 1$, then $[x_0 : x_1 : x_2]$ and $[y_0 : y_1 : y_2]$ ***are equivalent***.

We call the class of all equivalent parametrizations the place of a curve.

Definition 6

If $x \in K[[t^r]]$ for some $r > 1$, then x ***is reducible*** as we can simplify x by replacing t^r by s.

the Newton-Puiseux Method

criterion on reducibility

Theorem 5

The parametrization $(x, y) \in K[[t]]$, *with* $x = t^n$, $n > 0$, *and* $y = a_1 t^{n_1} + a_2 t^{n_2} + \cdots$, $0 < n_1 < n_2 < \cdots$, $a_i \neq 0$ *is reducible* $\Leftrightarrow \gcd(n, n_1, n_2, \ldots) > 1$.

Proof. $\Leftarrow$ If $\gcd(n, n_1, n_2, \ldots) = r > 1$, then replace t^r by s.

$\Rightarrow$ If reducible, there is an $s \in K[[t]]$, with $O(s) = 1$ such that $x(s), y(s) \in K[[t^r]]$, with $r > 1$.

Suppose $s/t \not\in K[[t^r]]$, then s is of the form

$$s = t(b_0 + b_1 t^{h_1} + \cdots + b_k t^{h_k} + \cdots)$$

and we will derive a contradiction ...

deriving a contradiction

Let $s = t(b_0 + b_1 t^{h_1} + \cdots + b_k t^{h_k} + \cdots)$, where $b_0 b_k \neq 0$ and furthermore: $r \nmid h_k$, assuming what we want to prove were not true.

$$\begin{aligned} x(s) = s^n &= t^n(b_0 + b_1 t^{h_1} + \cdots + b_k t^{h_k} + \cdots)^n \\ &= t^n(b_0 + b_1 t^{h_1} + \cdots)^n \\ &\quad + n b_k t^{n+h_k}(b_0 + b_1 t^{h_1} + \cdots)^{n-1} + \cdots \end{aligned}$$

Since $x(s)$ is reducible, $x(s) \in K[[t^r]]$, so $r|n$ and because $x(s)$ starts with $t^n b_0$:

$$x(s) - t^n(b_0 + b_1 t^{h_1} + \cdots)^n = n t^{n+h_k} b_k b_0^{n-1} + \cdots$$

As the left of the equation above belongs to $K[[t^r]]$, also the right hand side belongs to $K[[t^r]]$ and therefore $r|n + h_k$ which contradicts $r \nmid h_k$. Hence $s = tz$, $z \in K[[t^r]]$.

proof continued

Let us now look at y. Suppose at least one of the $n_1, n_2, \ldots$ is not divisible by r and let n_k be the first k such that $r \not| n_k$.

Consider $y(s)$, substituting t in y by tz:

$$\begin{aligned} & y(s) - (a_1 t^{n_1} z^{n_1} + a_2 t^{n_2} z^{n_2} + \cdots) \\ = \quad & a_{n_k} t^{n_k} (b_0 + b_1 t^{h_1} + \cdots)^{n_k} + \cdots \\ = \quad & a_{n_k} b_0^{n_k} t^{n_k} + \cdots \end{aligned}$$

As the left of the equation above belongs to $K[[t^r]]$, the right cannot belong to $K[[t^r]]$ unless the assumption $r \not| n_k$ is wrong. □

normal form

Theorem 6

In a suitable coordinate system, any given parametrization is equivalent to one of the type

$$\begin{cases} x = t^n, & 0 < n \\ y = a_1 t^{n_1} + a_2 t^{n_2} + \cdots, & 0 < n_1 < n_2 < \cdots. \end{cases}$$

Proof. Choose the center as the origin of the coordinate system, then

$$x_1 = t^n(b_0 + b_1 t + \cdots), n > 0, \quad y_1 = t^{n_1}(c_0 + c_1 t + \cdots), n_1 > 0.$$

proof continued

We may assume $b_0 \neq 0$, otherwise we relable, increasing n.

Let $s = t(d_1 + d_2 t + \cdots)$, $d_1 \neq 0$ and consider $x = x_1(s)$, $y = y_1(s)$, replacing t by s in x:

$$\begin{aligned} x &= t^n(d_1 + d_2 t + \cdots)^n (b_0 + b_1(d_1 t + \cdots) + \cdots) \\ &= t^n(d_1^n b_0 + n(d_1^{n-1} d_2 b_0 + d_1^{n+1} b_1)t + \cdots \\ &\quad + n(d_1^{n-1} d_i b_0 + P_i(b_1, \ldots, b_i, d_1, \ldots, d_{i-1}))t^i + \cdots). \end{aligned}$$

We compute d_1, d_2, …, so that $x = t^n$:

$$d_1^n = b_0^{-1}, d_2 = -(n d_1^{n-1} b_0)^{-1} d_1^{n+1} b_1,$$

$$d_i = -(n d_1^{n-1} b_0)^{-1} P_i, i = 3, 4, \ldots.$$

Then we have $x = t^n$ and (x, y) is of the required type. □

the Newton-Puiseux Method

fractional power series

Instead of writing the parametrizations as $(t^n, y(t))$,
we write (t, y), where $y \in K[[x^{1/n}]]$.

Symbolically, we manipulate $x^{1/n}$ along the following rules:
$x^{1/1} = x$, $x^{m/n} = (x^{1/n})^m$, $(x^{1/(kn)})^k = x^{1/n}$, $x^{km/(kn)} = x^{m/n}$.

Since $(x^{1/(kn)})^k = x^{1/n}$, we have $K[[x^{1/n}]] \subset K[[x^{1/(kn)}]]$.

Definition 7

$K[[x]]^* = \bigcup_{n=1}^{\infty} K[[x^{1/n}]]$ is the set of ***fractional power series***.

For $x \in K[[x^{1/n}]]$ and $y \in K[[x^{1/m}]]$, then $x, y \in K[[x^{1/(mn)}]]$
and so are their sum, product, and quotient.
Thus, $K[[x]]^*$ is a field.

the theorem of Puiseux

Theorem 7 (Puiseux)

If K is algebraically closed,
then $K[[x]]^*$ is algebraically closed.

Corollary

For any $f \in K[[x]]^*[y]$, there is a $z \in K[[x]]^*$: $f(z) = 0$,
or more explicitly:

$$\begin{aligned} f(y) &= \sum_{i=0}^{n} a_i y^i, \quad a_i \in K[[x]]^* \\ &= a_n \prod_{i=1}^{n} (y - z_i), \quad f(z_i) = 0. \end{aligned}$$

Constructive proof gives the Newton-Puiseux method.

the form of a root

What does a root of $f \in K[[x]]^*[y]$ look like?
$z = 0$ is trivial, we assume $a_0 \neq 0$.

A general solution has the form

$$y = c_1 x^{\gamma_1} + c_2 x^{\gamma_1+\gamma_2} + c_3 x^{\gamma_1+\gamma_2+\gamma_3} + \cdots, \ c_i \neq 0, \gamma_1 = O(y),$$

and $\gamma_2 > 0, \gamma_3 > 0, \ldots$. Abbreviating y to: $c = c_1$, $\gamma = \gamma_1$:
$y = x^\gamma(c + y_1)$, $y_1 \in K[[x]]^*$ and substituting in f leads to

$$\begin{aligned} f(y) &= a_0 + a_1 x^\gamma (c + y_1) + a_2 x^{\gamma 2} (c + y_1)^2 + \cdots \\ &\quad + a_n x^{\gamma n} (c + y_1)^n \\ &= a_0 + a_1 c x^\gamma + a_2 c^2 x^{2\gamma} + \cdots + a_n c^n x^{n\gamma} + g(y_1). \end{aligned}$$

As $y_1 = c_2 x_2^\gamma$, $O(y_1) = \gamma_2 > 0 \Rightarrow O(g) \geq \gamma_2$ and g collects terms of order higher than some of the $a_i c^i x^{i\gamma}$.

necessary conditions

Necessary conditions for $f(y) = 0$ are then:

1. At least two of $a_i c^i x^{i\gamma}$ have the same order, say j and k, and that order is less than any other:

$$\begin{array}{rcccl} O(a_j c^j x^{j\gamma}) & = & O(a_k c^k x^{k\gamma}) & \leq & O(a_i c^i x^{i\gamma}), \ i \neq j, i \neq k \\ O(a_j) + j\gamma & = & O(a_k) + k\gamma & \leq & O(a_i) + i\gamma. \end{array}$$

2. The coefficients of lowest order must cancel:

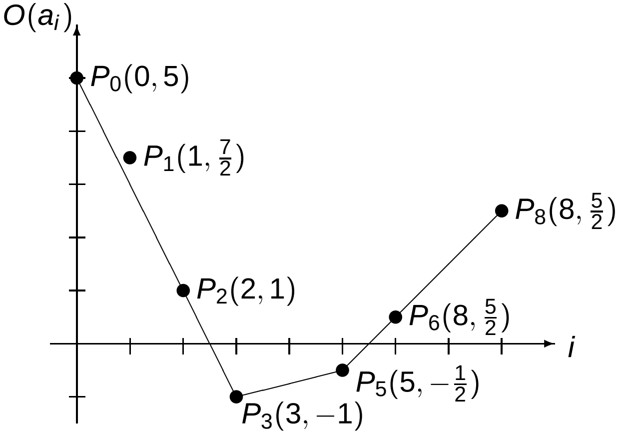
$$\sum_{\substack{\ell \\ O(a_\ell) + \ell\gamma = O(a_j) + j\gamma}} a_\ell c^\ell = 0.$$

To visualize the conditions on γ,
we use the Newton polygon.

the Newton-Puiseux Method

the Newton polygon

$$
\begin{aligned}
f(y) \quad = \quad & a_0x^5 + a_1x^{7/2}y + a_2xy^2 + a_3x^{-1}y^3 \\
& + a_5x^{-1/2}y^5 + a_6x^{1/2}y^6 + a_7x^{10/3}y^7 + a_8x^{5/2}y^8.
\end{aligned}
$$



slope conditions

For points $P_i(i, O(a_i))$, the order γ of the solution defines a slope $(\gamma, 1)$. With $(\gamma, 1)$ we can rank the points via the inner product: $\langle (i, O(a_i)), (\gamma, 1) \rangle = i\gamma + O(a_i)$.

This slope condition gives a geometric interpretation to the first necessary condition on the orders of the coefficients a_i.

In particular, the condition on γ is

$$O(a_j) + j\gamma = O(a_k) + k\gamma \quad \Rightarrow \quad \gamma = \frac{O(a_j) - O(a_k)}{k - j}.$$

slopes for the example

We have three slopes:

$$P_0P_2P_3 \quad : \quad \gamma = \frac{O(a_0) - O(a_2)}{2 - 0} = \frac{5-1}{2} = 2, \\ a_0c^0 + a_2c^2 + a_3c^3 = 0$$

$$P_3P_5 \quad : \quad \gamma = \frac{O(a_3) - O(a_5)}{5 - 3} = \frac{-1-(-1/2)}{3} = -1/4, \\ a_3c^3 + a_5c^5 = 0$$

$$P_5P_6P_8 \quad : \quad \gamma = \frac{O(a_5) - O(a_6)}{6 - 5} = \frac{-1/2-1/2}{1} = -1, \\ a_5c^5 + a_6c^6 + c_8c^8 = 0.$$

There are 3 + 2 + 3 = 8 nonzero roots of the polynomials.

left to consider

Three issues still need careful consideration:

1. There are $\deg(f)$ values for c.

   We look at consecutive points on the lower hull of the Newton polygon. The conditions on the coefficients of lowest order are polynomials with exponents as points on the slopes of the Newton polygon.

2. Computation of other terms in the expansion.

   Consider $y = x^{\gamma_1}(c_1 + c_2 x^{\gamma_2})$.
   Either $y = c_1 x^{\gamma_2}$ and we are done,
   or we substitute and look for positive slopes, as $\gamma_2 > 0$.

3. The powers γ's have bounded denominators.

Summary + Exercises

The theorem of Puiseux is a generalization of the fundamental theorem of algebra. The constructive proof with the Newton polygon leads to the Newton-Puiseux method to develop power series solutions.

Exercises:

1. Similar to deriving polynomials, we can take derivatives of power series. Show that the product rule for the derivation of power series holds.
2. Maple offers the procedure `algcurves[puiseux]` to determine the Puiseux expansion of an algebraic function. Apply this procedure to the example used to illustrate the Newton polygon.

more exercises

3. Take one root of the $f(y) = 0$ in the example used to illustrate the Newton polygon and compute the second term in its series expansion. Draw the Newton polygon of $f(y_1)$ after substitution of $y = x^{\gamma_1}(c_1 + c_2 y_1^{\gamma_2})$ and verify there is a positive choice for γ_2. Is this choice unique?
4. Continue the previous exercise by computing more terms of the series expansion. Verify that the denominators of the γ's are bounded.