

Quotient Rings

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- 2 Design of Experiments
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- 3 the Quotient Algebra
monomials under the staircase
- 4 the Buchberger-Möller algorithm
computing the monomials under the staircase
using Macaulay 2

MCS 563 Lecture 8
Analytic Symbolic Computation
Jan Verschelde, 28 January 2011

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defining ideals

Our point of departure is not a polynomial system, but a set of points $V \subset \mathbb{C}^n$.

The defining ideal of V is the set of all polynomials which vanish on V , or formally:

$$I(V) = \{ f \in \mathbb{C}[\mathbf{x}] \mid f(\mathbf{z}) = \mathbf{0}, \forall \mathbf{z} \in V \}.$$

If V contains only one point $\mathbf{z} = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$, $V = \{\mathbf{z}\}$, then $I(V) = \langle x_1 - z_1, x_2 - z_2, \dots, x_n - z_n \rangle$.

For any ideal I , the quotient ring is

$$\mathbb{C}[\mathbf{x}]/I = \{ r \in \mathbb{C}[\mathbf{x}] \mid f \rightarrow_I r, \forall f \in \mathbb{C}[\mathbf{x}] \}.$$

For $I = I(\{\mathbf{z}\})$ we see that the remainder after division by the polynomials $x_i - z_i$, $i = 1, 2, \dots, n$, will always be a constant. Thus we may identify $\mathbb{C}[\mathbf{x}]/I(\{\mathbf{z}\})$ with \mathbb{C} .

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Let $V = \{\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_k\}$ with $\mathbf{z}_i \neq \mathbf{z}_j$ for $i \neq j$.

We denote by $\mathbf{z}_{i,j}$ the j th coordinate of \mathbf{z}_i .

If all points in V have a *different first* coordinate $z_{i,1}$, then the Lagrange polynomials in x_1 separate the points.

The i th Lagrange polynomial is

$$L_i(\mathbf{x}) = \prod_{\substack{j=1 \\ j \neq i}}^n \left(\frac{x_1 - z_{j,1}}{z_{i,1} - z_{j,1}} \right).$$

We have

$$L_i(\mathbf{z}_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

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interpolating polynomials

A polynomial $s_{\mathbf{z}_i}(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]$ is a separator for the point \mathbf{z}_i if

$$s_{\mathbf{z}_i}(\mathbf{z}_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Given a value v_i for each point $\mathbf{z}_i \in V$, a polynomial $p \in \mathbb{C}[\mathbf{x}]$ is interpolating the values v_i at the points \mathbf{z}_i if

$$p(\mathbf{z}_i) = v_i, \quad \text{for } i = 1, 2, \dots, k$$

Given a separators for the points,

$$p(\mathbf{x}) = \sum_{i=1}^k v_i s_{\mathbf{z}_i}(\mathbf{x})$$

satisfies the k interpolation conditions.

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As a special case of the interpolating polynomials, consider the polynomials

$$p_j(\mathbf{x}) = x_j - \sum_{i=1}^k z_{i,j} s_{z_i}(\mathbf{x}), \quad j = 1, 2, \dots, n,$$

where the values v_i are simply the coordinates of the points.

Evaluating p_j at \mathbf{z}_ℓ , we see that

$$p_j(\mathbf{z}_\ell) = z_{\ell,j} - \sum_{i=1}^k z_{i,j} s_{z_i}(\mathbf{z}_\ell) = z_{\ell,j} - z_{\ell,j} s_{z_\ell}(\mathbf{z}_\ell) = 0.$$

So we have polynomials vanishing at the set V .

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If all points have a distinct first coordinate, then with Lagrange polynomials as separators we construct a lexicographic Gröbner basis for $I(V)$:

$$g = \left\{ \begin{array}{l} x_n - \sum_{i=1}^k z_{i,n} L_i(\mathbf{x}), \dots, x_2 - \sum_{i=1}^k z_{i,2} L_i(\mathbf{x}), \\ (x_1 - z_{1,1})(x_1 - z_{2,1}) \cdots (x_1 - z_{k,1}) \end{array} \right\}.$$

By its shape, the solution set equals V .

For any $f \in \mathbb{C}[\mathbf{x}]$, we have $f \rightarrow_g r$ with

$$r = r_0 + r_1 x_1 + r_2 x_1^2 + \cdots + r_{k-1} x_1^{k-1}, \text{ so: } \dim(\mathbb{C}[\mathbf{x}]/\langle g \rangle) = k.$$

If some of the points in V have the same first coordinate, then we may apply a random linear coordinate change.

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5 factors: color, shape, weight, material, and price determine the choice of features in a new product. Assume 3 choices for every factor: $3^5 = 243$ prototypes. Test consumer reaction: need to find a subset of 3^5 .

Factorial design: determine a subset \mathcal{F} of all choices from which we construct a good model to rate the product.

For example: if the 5 features are color (C), shape (S), weight (W), material (M), and price (P), then a model is a polynomial $f \in \mathbb{R}[C, S, W, M, P]$.

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- Let $\mathcal{F} := \{\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_s\} \subset \mathcal{D}$. What are the complete linear models which can be identified by \mathcal{F} ?
 → what are the possible bases of $\mathbb{C}[\mathbf{x}]/I(\mathcal{F})$?
- Let the polynomial $f(\mathbf{x})$ be a complete linear polynomial model, where the support of f is a subset of the normal set of \mathcal{D} . What are the minimal fractions which identify the model? → for a set O of monomials, find \mathcal{F} : O is a basis of $\mathbb{C}[\mathbf{x}]/I(\mathcal{F})$
- What are the fractions \mathcal{F} of \mathcal{D} , with $\#\mathcal{F} < \#\mathcal{D}$ which identify the highest (or lowest) number of complete linear polynomial models? → find O with highest (or lowest) $\#O$, O is basis for $\mathbb{C}[\mathbf{x}]/I(\mathcal{F})$.

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If $\#V = k$ isolated points for the ideal $I = I(V)$, then the quotient ring $\mathbb{C}[\mathbf{x}]/I$ is a k -dimensional vector space, called the quotient algebra.

A Gröbner basis with the total degree order ($x > y$):

$$g = \{ 2x^2 + 3xy + y^2 - 3x - 3y, xy^2 - x, y^3 - y \}.$$

The initial ideal is generated by $\text{LT}(g) = \{x^2, xy^2, y^3\}$.

The only monomials not in the ideal are $O = \{1, x, y, xy, y^2\}$ and therefore:

$$\forall p \in \mathbb{C}[x, y] : p \rightarrow_g r = c_1 + c_2x + c_3y + c_4xy + c_5y^2$$

for some coefficients $c_i, i = 1, 2, \dots, 5$. In Maple we type:
`Groebner [NormalForm] (x^2 * y, g, tdeg(x, y)) ;`

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multiplication matrices

Turning the set O into a basis vector B using \rightarrow_g on xB , we define an eigenvalue problem $xB = M_x B$:

$$x \begin{bmatrix} 1 \\ x \\ y \\ xy \\ y^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{3}{2} & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{3}{2} & -\frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ xy \\ y^2 \end{bmatrix}$$

where the matrix M_x defines the multiplication by $x \bmod I$.

The multiplication by $y \bmod I$ defines M_y .

The eigenvalues of M_x and M_y give respectively the x and y -coordinates of the solutions.

So the multiplication matrices generalize the familiar companion matrices for polynomials in one variable.

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Theorem

Consider $f(\mathbf{x}) = \mathbf{0}$ with $V = f^{-1}(\mathbf{0}) \subset \mathbb{C}^n$.

Assume that all solutions have multiplicity one. Then:

$$\#V < \infty \quad \Leftrightarrow \quad \dim(\mathbb{C}[\mathbf{x}]/\langle f \rangle) = \#V.$$

\Rightarrow : the Shape lemma gives a lexicographic Gröbner basis.
The size of basis of the quotient ring is $\#V$.

\Leftarrow : we define the multiplication matrices to compute the
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Below is a simple version of the central theorem in Numerical Polynomial Algebra of Hans J. Stetter.

Theorem

Let I be a zero dimensional ideal with $\#V(I) = k$, $V(I) \subset \mathbb{C}^n$. Then there is a commuting family of multiplication matrices M_{x_i} , each of dimension k , for $i = 1, 2, \dots, n$. The multiplication matrices have a joint invariant subspace. Each joint eigenvector corresponds to one solution.

This central theorem establishes the link between zero dimensional polynomial ideals and eigenvalue problems.

multiple solutions

$$f(\mathbf{x}) = \begin{cases} x_1 + x_2 + x_3 - 1 = 0 \\ \frac{1}{5}x_1^3 + \frac{1}{2}x_2^2 - x_3 + \frac{1}{2}x_3^2 + \frac{1}{2} = 0 \\ x_1 + x_2 + \frac{1}{2}x_3^2 - \frac{1}{2} = 0. \end{cases}$$

With Maple:

```
[> gb := Groebner[gbasis](P, tdeg(x[1], x[2], x[3]));
[> ns, rv
:= Groebner[SetBasis](gb, tdeg(x[1], x[2], x[3]));
```

The last command returns the normal set as a list in `ns` and as a table in `rv`: $[1, x_3, x_2, x_2x_3, x_2^2, x_2^2x_3]$.

The size of the normal set tells us that the system has six solutions, *counted with multiplicities*.

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```
[> Mx[1] := Groebner[MulMatrix]
      (x[1], ns, rv, gb, tdeg(x[1], x[2], x[3]));
 [> v1 := LinearAlgebra[Eigenvalues](Mx[1]);
```

The eigenvalues of M_{x_1} are $[-\frac{5}{2}, -\frac{5}{2}, 0, 0, 0]$.

Two distinct solutions: $(-\frac{5}{2}, -\frac{5}{2}, 1)$ and $(0, 0, 1)$, respectively with multiplicities two and four.

```
[> J := LinearAlgebra[JordanForm](Mx[1]);
```

$$J = \left[\begin{array}{cc|cccc} -\frac{5}{2} & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{5}{2} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

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A set of monomials O is called a *normal set* when for all $\mathbf{x}^{\mathbf{a}} \in O$: if $\mathbf{x}^{\mathbf{b}}$ divides $\mathbf{x}^{\mathbf{a}}$, then $\mathbf{x}^{\mathbf{b}} \in O$.

For all generators $\mathbf{x}^{\mathbf{a}}$ of $\langle \text{LT}(I(V)) \rangle$:

$$\mathbf{x}^{\mathbf{a}} = x_i \mathbf{x}^{\mathbf{b}} \quad \text{for some } i \text{ and } \mathbf{x}^{\mathbf{b}} \in O.$$

Note: $\dim(\mathbb{C}[\mathbf{x}]/I(V)) = \#V = \#O$.

Specifications for the algorithm:

Input: $V = \{\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_k\}$ and a term order $<$.

Output: g is a Gröbner basis of $I(V)$ with respect to $<$;

O is a normal set, $\#O = k$;

S is a set of separators for the points in V .

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$g := \emptyset; O := \emptyset; S := \emptyset; r := 0; L := \{1\};$

while $L \neq \emptyset$ do

$$T := \min_{<}(L); L := L \setminus T; f := T - \sum_{i=1}^k T(\mathbf{z}_{\pi(i)})s_i;$$

if f vanishes on V then

$$g := g \cup \{f\};$$

$$L := L \setminus \{ \text{multiples of } T \};$$

else

$$O := O \cup T; r := r + 1;$$

$$\pi(r) := \min\{ i \mid f(\mathbf{z}_i) \neq 0 \};$$

$$s_r := f/f(\mathbf{z}_{\pi(r)}); S := S \cup \{s_r\};$$

for i from 1 to $r - 1$ do

$$s_i := s_i - s_i(\mathbf{z}_{\pi(r)})s_r;$$

end for;

$$L := L \cup \{ x_i T \mid i = 1, 2, \dots, n \} \setminus \text{LT}(g);$$

end if;

end while.

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$s_r := f/f(\mathbf{z}_{\pi(r)}); S := S \cup \{s_r\};$

for i from 1 to $r - 1$ do

$s_j := s_j - s_j(\mathbf{z}_{\pi(r)})s_r;$

end for;

$L := L \cup \{ x_i T \mid i = 1, 2, \dots, n \} \setminus \text{LT}(g);$

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$$s_i := s_i - s_i(\mathbf{z}_{\pi(r)})s_r;$$

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end if;

end while.

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An implementation of the Buchberger-Möller algorithm
is available in Macaulay 2 in the package `Points`:

```
$ M2
```

```
Macaulay2, version 1.3.1
```

```
with packages: ConwayPolynomials, Elimination, Inte  
PrimaryDecomposition, ReesAlgebra, S
```

```
i1 : loadPackage "Points";
```

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```
i2 : M = matrix{{1,2,3},{4,5,6}}
```

```
o2 = | 1 2 3 |
      | 4 5 6 |
      2      3
```

```
o2 : Matrix ZZ <--- ZZ
```

```
i3 : R = QQ[x,y,MonomialOrder=>Lex];
```

```
i4 : (Q,inG,G) = points(M,R)
```

```
o4 = ({1, y, y^2}, ideal (y^3, x),
```

```
      {y^3 - 15y^2 + 74y - 120, x - y + 3})
```

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```
i5 : M2 = matrix{{1,2,3},{2,2,2}}
```

```
o5 = | 1 2 3 |
      | 2 2 2 |
```

```
                2          3
o5 : Matrix ZZ  <--- ZZ
```

```
i6 : (Q2,inG2,G2) = points(M2,R)
```

```
                2          3
o6 = ({1, x, x }, ideal (y, x ),
```

```
                3          2
      {y - 2, x  - 6x  + 11x - 6})
```

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```
i7 : M3 = matrix{{1,1,2},{1,2,2}}
```

```
o7 = | 1 1 2 |
      | 1 2 2 |
```

```
o7 : Matrix ZZ  $\begin{matrix} & & 2 & & 3 \\ & & & & \\ & & & & \end{matrix}$  <--- ZZ
```

```
i8 : (Q3,inG3,G3) = points(M3,R)
```

```
o8 = ({1, y, x}, ideal (y2, x*y2, x2),
```

```
{y2 - 3y + 2, x*y2 - 2x - y + 2, x2 - 3x + 2})
```

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The Shape Lemma and the Buchberger-Möller algorithm allow to define ideals given a finite set of points.

Exercises:

- 1 For two finite sets V and W , show that $V \subset W$ implies $I(V) \supset I(W)$.
- 2 Use the definition of the Gröbner basis to show that the set of polynomials defined by the Shape lemma is a Gröbner basis for $I(V)$.
- 3 Consider $V = \{(1/3, 5, 1/2), (1, 2, 1/5), (2, 2, 3), (2, 2, 2)\}$. Observe that the third coordinate is different for each point in V . Use this observation to compute the shape lemma representation of V .

more exercises

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- 4 For the set V as in the previous exercise, take a lexicographic term order and run the Buchberger-Möller algorithm.
- 5 Download the free computer algebra system CoCoA from <http://cocoa.dima.unige.it/> and use it to solve the previous exercises. Alternatively, you may also use Macaulay 2.

one last exercise

6 Consider the ideal

$$I = \langle xy + z - xz, x^2 - z, 2x^3 - x^2yz - 1 \rangle \text{ over } \mathbb{Q}[x, y, z].$$

Use CoCoA, Maple, Macaulay 2, or Sage for the following:

- 1 Compute a Gröbner basis with respect to the lexicographic order $x > y > z$. How many points does $V(I)$ have? Give the coordinates for all points in $V(I)$.
- 2 Use $V(I)$ to bring the ideal in the shape lemma and compare with the lexicographic Gröbner basis.
- 3 Use the total degree order to compute a Gröbner basis g for I . What is the normal set and corresponding basis vector B for the quotient ring? Apply the division algorithm \rightarrow_g to define M_x . Compute the eigenvalues for M_x and verify that they give the x -coordinates of the points in $V(I)$.

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