Review and Summary of the first part of MCS 563

Our object of study is a system $f(\mathbf{x}) = \mathbf{0}$ of N polynomials $f = (f_1, f_2, \ldots, f_N)$, with complex coefficients: $f_i \in \mathbb{C}[\mathbf{x}]$, $i = 1, 2, \ldots, N$, in n variables: $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. We are interested in algorithms to compute and describe the solution set $V = f^{-1}(\mathbf{0})$.

The topics are summarized in the titles of the lectures:

1/10	Lec 1	introduction (Newton and Bézout)	1/12	Lec 2	elimination methods (resultants)
1/14	Lec 3	homotopies and predictor-corrector	1/19	Lec 4	rewriting polynomials (division algorithm)
1/21	Lec 5	alpha theory to certify roots	1/24	Lec 6	Gröbner bases (Buchberger's algorithm)
1/26	Lec 7	multihomogenization (linear-product)	1/28	Lec 8	quotient rings (shape lemma)
1/31	Lec 9	Condition and Scaling	2/04	Lec 10	Gröbner basis conversion (FGLM)
2/06	Lec 11	coefficient-parameter or cheater	2/09	Lec 12	Rational Univariate Representation
2/11	Lec 13	Real Homotopies (turning points)	2/14	Lec 14	Kronecker Parametrization (Newton-Hensel)
2/16	Lec 15	the Newton-Puiseux method			
2/18	Lec 16	Kushnirenko's theorem (Newton polytopes)			
2/21	Lec 17	mixed volumes (Cayley trick)			
2/23	Lec 18	Bernshteĭn's second theorem			
2/25	Lec 19	Polyhedral Homotopies			

We can classify the topics in three categories:

Numerical Homotopy Continuation: Starting in the first lecture with Newton's method, homotopies were introduced in lecture three. Every odd numbered lecture from 3 till 13 considered polynomial systems from a numerical perspective.

Symbolic Rewriting and Elimination: In lecture 2 we used resultants to introduce the main theorem of elimination theory. In every even numbered lecture from 2 till 14, we looked at polynomial systems from a symbolic perspective, using term rewriting techniques as solution methods.

Polyhedral Methods: Newton polytogons arose in lecture 15 in a method to compute Puiseux series. In the last four lectures, we stated the theorems of Kushnirenko and Bernshteĭn, ending with polyhedral homotopies.

Continuing the classification of materials in a tripartite fashion, we distinguish concepts, theorems, and algorithms. The exercises usually provide more examples of definitions encountered in the lectures and give opportunities to explore the algorithms.

A typical question we ask when we see a polynomial system is to bound the number of isolated solutions. We have seen two generalizations of Bézout's theorem and used mixed volumes to compute a root count. With every root count we associate a start system to be used in a homotopy.

An alternative look at a polynomial system applies elimination, either explicitly via resultants or the shape lemma, or more implicitly via eigenvalue problems and rational univariate parameterization. The main tool here is a Gröbner basis which turns the division algorithm into a normal form algorithm allowing computations in the quotient ring.