Rational Univariate Representation

1. Stickelberger’s Theorem
   - a rational univariate representation (RUR)

2. The Elbow Manipulator
   - a spatial robot arm with three links

3. Application of the Newton Identities
   - the Newton identities
   - computing the characteristic polynomial
   - computing the coordinates

4. Algorithm and Software
   - pseudo code to compute a RUR, using Maple

MCS 563 Lecture 12
Analytic Symbolic Computation
Jan Verschelde, 10 February 2014
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A Rational Univariate Representation (or RUR) is

\[ R = \left\{ p_0(T) = 0, \ x_i = \frac{p_i(T)}{q(T)}, i = 1, 2, \ldots, n \right\} \]

where \( p_0, p_1, p_2, \ldots, p_n, q \in \mathbb{C}[T] \).

This set \( R \) represents the coordinates of the zeroes of a solution set \( V, \) \( \# V = D < \infty, \) of some system of polynomials in \( \mathbb{C}[x] \).

\( D \) is counted with multiplicities.

The number of distinct zeroes in \( V \) is denoted by \( d \).

RUR generalizes the Shape Lemma representation.
Let the linear form $L(x)$ separate the zeroes:

$$L(z_i) \neq L(z_j), \text{ for } z_i, z_j \in V, i \neq j.$$ 

Consider the multiplication map

$$m_L : \mathbb{C}[x]/I(V) \to \mathbb{C}[x]/I(V) : h \mapsto \left((h \cdot L) \rightarrow_{G_>} r\right)$$

where $\rightarrow_{G_>}$ represents the normal form algorithm implemented by the division algorithm using some Gröbner basis $G_>$. 
choosing a separator

**Lemma**

If $V$ has $d$ distinct zeroes, at least one of the

$$u_i(x_1, x_2, x_3, \ldots, x_n) = x_1 + ix_2 + i^2x_3 + \cdots + i^{n-1}x_n,$$

for

$$0 \leq i \leq (n-1) \binom{d}{2}$$

is separating, i.e.: $\forall z_j, z_k \in V, j \neq k, u_i(z_j) \neq u_i(z_k)$. 
proof of the lemma

Proof. For the pair \((z_j, z_k), j \neq k\), of two distinct zeroes in \(V\) with components \(z_j = (z_{j1}, z_{j2}, \ldots, z_{jn})\) \(z_k = (z_{k1}, z_{k2}, \ldots, z_{kn})\), consider the bad situation when \(u_t(z_j) = u_t(z_k)\), corresponding to

\[ p(t) = (z_{j1} - z_{k1}) + (z_{j2} - z_{k2})t + \cdots + (z_{jn} - z_{kn})t^{n-1}. \]

Because \(z_j \neq z_k\), \(p \neq 0\) and therefore \(p\) can have at most \(n - 1\) zeroes.

So for each pair of zeroes of \(V\) we have at most \(n - 1\) bad choices for \(u_i\) and the number of pairs of zeroes is \(d(d - 1)/2\), yielding a total of at most \((n - 1)d(d - 1)/2\) nonseparating \(u_i\)'s.

But the set of \(u_i\)'s consists of \((n - 1)d(d - 1)/2 + 1\) elements, so there is at least one separating \(u_i\). \(\square\)
The multiplication map $m_L$ is a linear map with matrix $M_L$. The eigenvalues of $M_L$ give values for $L(z)$, for all $z \in V$, occurring with the same multiplicity $\mu_z$.

As a consequence, the characteristic polynomial of $M_L$ is

$$p_0(T) = \det(M_L - T I_D) = \prod_{z \in V} (T - L(z))^{\mu_z},$$

with $\mu_z$ the multiplicity of the root. The trace of $M_L$ is

$$\text{trace}(M_L) = \sum_{z \in V} \mu_z L(z).$$

The determinant of $M_L$ is

$$\det(M_L) = \prod_{z \in V} L(z)^{\mu_z}.$$
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The Elbow Manipulator

We consider a spatial robot arm with three links.
inverse kinematics

We consider a spatial robot arm with three links.

The input to an inverse position problem is

\[
\begin{pmatrix}
  n_x & o_x & a_x & p_x \\
  n_y & o_y & a_y & p_y \\
  n_z & o_z & a_z & p_z \\
  0 & 0 & 0 & 1
\end{pmatrix}
\]

representing position and orientation of the robot hand:

- \( p = (p_x, p_y, p_z) \) is the position of the hand;
- \( n = (n_x, n_y, n_z), \|n\|_2 = 1 \), normal;
- \( o = (o_x, o_y, o_z), \|o\|_2 = 1 \), orientation;
- \( a = (a_x, a_y, a_z), \|a\|_2 = 1 \), approach vector;

related by the cross product: \( n = o \times a \).
coordinate transformations

Wanted: angles $\theta_i$, $i = 1, 2, \ldots, 6$, $s_i = \sin(\theta_i)$, $c_i = \cos(\theta_i)$, and $s_i^2 + c_i^2 = 1$. Lengths of links are $L_2$, $L_3$, and $L_4$.

\[
\begin{pmatrix}
c_1 & 0 & s_1 & 0 \\
s_1 & 0 & -c_1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\cdot
\begin{pmatrix}
c_2 & -s_2 & 0 & c_2L_2 \\
s_2 & c_2 & 0 & s_2L_2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\cdot
\begin{pmatrix}
c_3 & -s_3 & 0 & c_3L_3 \\
s_3 & c_3 & 0 & s_3L_3 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\cdot
\begin{pmatrix}
c_4 & 0 & -s_4 & c_4L_4 \\
s_4 & 0 & c_4 & s_4L_4 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\cdot
\begin{pmatrix}
c_5 & 0 & s_5 & 0 \\
s_5 & 0 & -c_5 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\cdot
\begin{pmatrix}
c_6 & -s_6 & 0 & 0 \\
s_6 & c_6 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
= 
\begin{pmatrix}
n_x & o_x & a_x & p_x \\
n_y & o_y & a_y & p_y \\
n_z & o_z & a_z & p_z \\
0 & 0 & 0 & 1 \\
\end{pmatrix},
\]
RUR and the Newton Identities

To compute

$$R = \left\{ p_0(T) = 0, \ x_i = \frac{p_i(T)}{q(T)}, i = 1, 2, \ldots, n \right\}$$

we proceed along two steps:

1. apply Newton’s formula for $p_0$;
2. compute in the quotient algebra to find the rest.
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the elementary symmetric polynomials

A polynomial $p$ in one variable $x$ defined by its roots $x_1, x_2, x_3, \text{ and } x_4$, written as a monic polynomial:

$$p(x) = (x - x_1)(x - x_2)(x - x_3)(x - x_4)$$

$$= x^4 - (x_1 + x_2 + x_3 + x_4)x^3$$

$$+ (x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4)x^2$$

$$- (x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4)x$$

$$+ x_1x_2x_3x_4$$

$$= x^4 - e_1(x_1, x_2, x_3, x_4)x^3 + e_2(x_1, x_2, x_3, x_4)x^2$$

$$- e_3(x_1, x_2, x_3, x_4)x + e_4(x_1, x_2, x_3, x_4).$$

The polynomials $e_1, e_2, e_3, \text{ and } e_4$ are the elementary symmetric polynomials.
the power sums

Substituting the roots \( x_1, x_2, x_3, \) and \( x_4 \) into \( p(x) = x^4 - e_1 x^3 + e_2 x^2 - e_3 x + e_4 \) gives

\[
0 = p(x_1) = x_1^4 - e_1 x_1^3 + e_2 x_1^2 - e_3 x_1 + e_4, \\
0 = p(x_2) = x_2^4 - e_1 x_2^3 + e_2 x_2^2 - e_3 x_2 + e_4, \\
0 = p(x_3) = x_3^4 - e_1 x_3^3 + e_2 x_3^2 - e_3 x_3 + e_4, \\
0 = p(x_4) = x_4^4 - e_1 x_4^3 + e_2 x_4^2 - e_3 x_4 + e_4.
\]

Adding up: \( 0 = s_4 - e_1 s_3 + e_2 s_2 - e_3 s_1 + 4e_4, \)

where \( s_1, s_2, s_3, \) and \( s_4 \) are the power sums:

\[
s_1 = x_1 + x_2 + x_3 + x_4, \\
s_2 = x_1^2 + x_2^2 + x_3^2 + x_4^2, \\
s_3 = x_1^3 + x_2^3 + x_3^3 + x_4^3, \\
s_4 = x_1^4 + x_2^4 + x_3^4 + x_4^4.
\]
the Newton Identities

Expressing the power sums in terms of the elementary symmetric polynomials:

\[ s_1 = e_1, \]
\[ s_2 = e_1 s_1 - 2e_2, \]
\[ s_3 = e_1 s_2 - e_2 s_1 + 3e_3, \]
\[ s_4 = e_1 s_3 - e_2 s_2 + e_3 s_1 - 4e_4. \]

The relations above allow the derivations of the power sums from the coefficients of a monic polynomial.
computing coefficients from power sums

Writing the Newton identities in another way:

\[ e_1 = s_1, \]
\[ 2e_2 = e_1 s_1 - s_2, \]
\[ 3e_3 = e_1 s_2 - e_2 s_1 + s_3, \]
\[ 4e_4 = e_1 s_3 - e_2 s_2 + e_3 s_1 - s_4, \]

we see that, given the power sums of the roots, we can derive the coefficients of the monic polynomial that vanishes at those roots.
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the characteristic polynomial

To derive the characteristic polynomial $p_0$, the trace of $L^i$ is $s_i = \sum_{z \in V} \mu_z L^i(z)$.

If $p_0(T) = \sum_{i=0}^{D} b_i T^{D-i}$, $D = \# V$, $b_i \in \mathbb{C}[x]$, then

$$\frac{p'_0(T)}{p_0(T)} = \sum_{z \in V} \frac{\mu_z}{T - L(z)} = \sum_{j \geq 0} \frac{\text{trace}(L^j)}{T^{j+1}}.$$ 

We have $p'_0(T) = \sum_{\ell=0}^{D-1} \sum_{j=0}^{D-\ell-1} \text{trace}(L^j) b_\ell T^{D-\ell-j-1}$.

Apply Newton's formula: $(D - i)b_i = \sum_{j=0}^{i} \text{trace}(L^j) b_{i-j}$.

This gives a linear system in the $b_i$'s.
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separating form $L$

For any $v \in A$, $A$ is the quotient algebra, we define

$$g_L(v, T) = \sum_{z \in V} \mu_z v(z) \prod_{y \in V(p_0), y \neq L(z)} (T - y).$$

Observe

$$\sum_{z \in V} \mu_z v(z) \prod_{y \in V(p_0), y \neq L(z)} (L(z) - y)$$

$$= \frac{g_L(v, L(z))}{g_L(1, L(z))} \sum_{z \in V} \mu_z \prod_{y \in V(p_0), y \neq L(z)} (L(z) - y) = v(z).$$

Then we let $v$ become a coordinate $x_i \ldots$
representing coordinates

As \( \frac{g_L(v, L(z))}{g_L(1, L(z))} = v(z) \), we let \( v = x_i \) and we have

\[
x_i = \frac{g_L(x_i, T)}{g_L(1, T)}, \quad i = 1, 2, \ldots, n.
\]

Note

\[
g_L(1, T) = \frac{p_0'(T)}{\text{GCD}(p_0'(T), p_0(T))}.
\]

If all roots occur with multiplicity one, the denominator \( g_L(1, T) \) is just the derivative of \( p_0(T) \).
computing $g_L(v, T)$

To compute $g_L(v, T)$, we define

$$\overline{p_0}(T) = \frac{p_0(T)}{\text{GCD}(p'_0(T), p_0(T))},$$

and

$$g_L(v, T) = \sum_{j=0}^{d-1} \sum_{k=0}^{d-j-1} \text{trace}(vL^j) a_i T^{d-j-k-1}, \quad \overline{p_0} = \sum_{i=0}^{d} a_i T^{d-i}.$$

Then we set $q(T) = g_L(1, T)$ and $p_i(T) = g_L(x_i, T)$ and obtain a representation for RUR.
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pseudo code

Input: \( G \), a Gröbner basis for an ideal \( I \) with term order \( > \),
\[
\# V(I) = D < \infty.
\]
Output: a rational univariate representation for \( V(\sqrt{I}) \).

compute \( \mathcal{N} > \) the basis vector for the quotient ring;
let \( D = \# \mathcal{N} > = \# V(I) \), counted with multiplicities;
compute \( \text{TrM} \) and deduce \( d = \# \text{distinct zeroes} \);
choose a separating element \( u \) as one of the \( u_i \)'s;
compute for \( m \) from 1 to \( D \): \( \text{trace}(u^m) \) and use \( u \) to form \( p_0(T) \);
compute \( \bar{p}_0 \) for \( \sqrt{I} \), if \( \deg(\bar{p}_0) < d \) then choose another \( u \);
for \( j \) from 1 to \( D \)
  for \( i \) from 0 to \( d \)
    compute \( \text{trace}(x_j u^i) \) and deduce \( g_u(x_j, T) \);
set \( q(T) = g_u(1, T) \) and \( p_i = g_u(x_i, T), i = 1, 2, \ldots, n \).
In Maple, we compute a RUR as follows:

\[
\begin{align*}
&> v := x[4],x[3],x[2],x[1]; \\
&> Groebner[Basis](f,plex(v)); \\
&> Groebner[RationalUnivariateRepresentation] \\
&\quad (f,v,output=factored);
\end{align*}
\]
Summary + Exercises

RUR generalizes the Shape Lemma representation.

Exercises:

1. Construct an example of a solution set $V$ in two variables and $\# V > 1$ where all except for one choice of the $u_i$'s fail to be separating.

2. Use a lexicographic term order to compute a Gröbner basis for the system Katsura, for $n = 3$ (see lecture 3). How many decimal places does the largest coefficient in this basis have? Compare with the size of the coefficients in the lecture note.

3. Use Maple's `Groebner[RationalUnivariateRepresentation]` on the example of the previous exercise.
Create a Maple worksheet to define the polynomial system for the elbow manipulator, for general choices of the position. Solve the system using the choices in the lecture note, either by Maple or by Sage/Singular.

Consider the system:

\[
\begin{align*}
24x_1x_2 - x_1^2 - x_2^2 - x_1^2x_2^2 - 13 &= 0 \\
24x_2x_3 - x_2^2 - x_3^2 - x_2^2x_3^2 - 13 &= 0 \\
24x_3x_1 - x_3^2 - x_1^2 - x_3^2x_1^2 - 13 &= 0.
\end{align*}
\]

Solve the system to verify that no variable is separating.

Find a separating element of the form as \( u_i \) an run through the steps of Algorithm to compute a RUR. Use a worksheet or a notebook to guide the computations.

Use the builtin commands in Maple or Sage to compute a rational univariate representation. Compare the output with the outcome of the step-by-step execution of Algorithm to compute a RUR.
Consider the following modification of the cyclic 5-roots problem:

\[
f(x) = \begin{cases} 
    x_1 + x_2 + x_3 + x_4 + x_5 = 0 \\
    x_1 x_2 + x_2 x_3 + x_3 x_4 + x_4 x_5 + x_5 x_1 = 0 \\
    x_1 x_2 x_3 + x_2 x_3 x_4 + x_3 x_4 x_5 + x_4 x_5 x_1 + x_5 x_1 x_2 = 0 \\
    x_2 x_3 x_4 + x_2 x_3 x_4 x_5 + x_3 x_4 x_5 x_1 + x_4 x_5 x_1 x_2 + x_5 x_1 x_2 x_3 = 0 \\
    x_1 x_2 x_3 x_4 x_5 - 1 = 0.
\end{cases}
\]

where the monomial \(x_1 x_2 x_3 x_4\) in the original cyclic 5-roots system is replaced by \(x_2 x_3 x_4\). Compute a Gröbner basis with the graded lexicographical order to determine the number of roots of this modified cyclic 5-roots system. Compute a rational univariate representation for this system. Compare the size of the coefficients between the lexicographical Gröbner basis (shape lemma) and the RUR.