Stickelberger's Theorem
a rational univariate representation (RUR)

The Elbow Manipulator
a spatial robot arm with three links

Newton Sums
computing the characteristic polynomial
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pseudo code to compute a RUR, using Maple

Rational Univariate Representation

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MCS 563 Lecture 12
Analytic Symbolic Computation
Jan Verschelde, 9 February 2011

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RUR

A Rational Univariate Representation (or RUR) is

$$R = \left\{ p_0(T) = 0,\ x_i = \frac{p_i(T)}{q(T)}, i = 1, 2, \ldots, n \right\}$$

where $p_0, p_1, p_2, \ldots, p_n, q \in \mathbb{C}[T]$.

This set R represents the coordinates of the zeroes of a solution set V, $\#V = D < \infty$, of some system of polynomials in $\mathbb{C}[\mathbf{x}]$. D is counted with multiplicities.

The number of distinct zeroes in V is denoted by d.

RUR generalizes the Shape Lemma representation.

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separating linear form

Let the linear form $L(\mathbf{x})$ separate the zeroes: $L(\mathbf{z}_i) \neq L(\mathbf{z}_j)$, for $\mathbf{z}_i, \mathbf{z}_j \in V$ $i \neq j$.

Consider the multiplication map

$$m_L : \mathbb{C}[\mathbf{x}]/I(V) \rightarrow \mathbb{C}[\mathbf{x}]/I(V) : h \mapsto ((h \cdot L) \rightarrow_{\mathcal{G}_>} r)$$

where $\rightarrow_{\mathcal{G}_>}$ represents the normal form algorithm implemented by the division algorithm using some Gröbner basis $\mathcal{G}_>$.

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choosing a separator

Lemma

If V has d distinct zeroes, at least one of the

$$u_i(x_1, x_2, x_3, \ldots, x_n) = x_1 + i x_2 + i^2 x_3 + \cdots + i^{n-1} x_n,$$

for

$$0 \leq i \leq (n-1) \binom{d}{2}$$

is separating, i.e.: $\forall \mathbf{z}_j, \mathbf{z}_k \in V, j \neq k, u_i(\mathbf{z}_j) \neq u_i(\mathbf{z}_k)$.

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proof of the lemma

Proof. For the pair $(\mathbf{z}_j, \mathbf{z}_k)$, $j \neq k$, of two distinct zeroes in V with components $\mathbf{z}_j = (z_{j1}, z_{j2}, \ldots, z_{jn})$ $\mathbf{z}_k = (z_{k1}, z_{k2}, \ldots, z_{kn})$ consider

$$p(t) = (z_{j1} - z_{k1}) + (z_{j2} - z_{k2})t + \cdots + (z_{jn} - z_{kn})t^{n-1}.$$

Because $\mathbf{z}_j \neq \mathbf{z}_k$, $p \neq 0$ and therefore p can have at most $n - 1$ zeroes. So for each pair of zeroes of V we have at most $n - 1$ bad choices for u_i and the number of pairs of zeroes is $d(d-1)/2$, yielding a total of at most $(n-1)d(d-1)/2$ nonseparating u_i's. But the set of u_i's consists of $(n-1)d(d-1)/2 + 1$ elements, so there is at least one separating u_i. □

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Stickelberger's Theorem

Theorem (Stickelberger's theorem)

The multiplication map m_L is a linear map with matrix M_L. The eigenvalues of M_L give values for $L(\mathbf{z})$, for all $\mathbf{z} \in V$, occurring with the same multiplicity $\mu_\mathbf{z}$.

As a consequence, the characteristic polynomial of M_L is

$$p_0(T) = \det(M_L - T I_D) = \prod_{\mathbf{z} \in V} (T - L(\mathbf{z}))^{\mu_\mathbf{z}},$$

with $\mu_\mathbf{z}$ the multiplicity of the root. The trace of M_L is

$$\mathrm{trace}(M_L) = \sum_{\mathbf{z} \in V} \mu_\mathbf{z} L(\mathbf{z}).$$

The determinant of M_L is $\det(M_L) = \prod_{\mathbf{z} \in V} L(\mathbf{z})^{\mu_\mathbf{z}}$.

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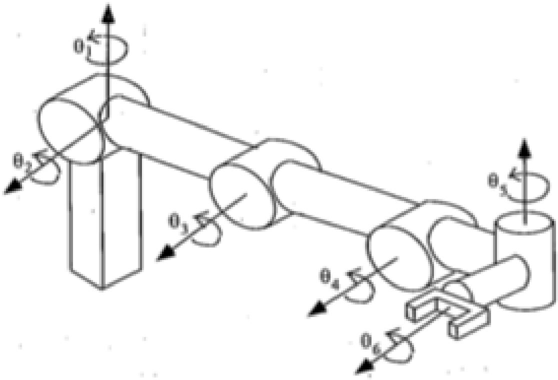
The Elbow Manipulator

We consider a spatial robot arm with three links.



inverse kinematics

We consider a spatial robot arm with three links.

The input to an inverse position problem is

$$\begin{pmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

representing position and orientation of the robot hand:

- $\mathbf{p} = (p_x, p_y, p_z)$ is the position of the hand;
- $\mathbf{n} = (n_x, n_y, n_z)$, $||\mathbf{n}||_2 = 1$, normal;
- $\mathbf{o} = (o_x, o_y, o_z)$, $||\mathbf{o}||_2 = 1$, orientation;
- $\mathbf{a} = (a_x, a_y, a_z)$, $||\mathbf{a}||_2 = 1$, approach vector;

related by the cross product: $\mathbf{n} = \mathbf{o} \times \mathbf{a}$.

coordinate transformations

Wanted: angles θ_i, $i = 1, 2, \ldots, 6$, $s_i = \sin(\theta_i)$, $c_i = \cos(\theta_i)$, and $s_i^2 + c_i^2 = 1$. Lengths of links are L_2, L_3, and L_4.

$$\begin{pmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} c_2 & -s_2 & 0 & c_2L_2 \\ s_2 & c_2 & 0 & s_2L_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\cdot \begin{pmatrix} c_3 & -s_3 & 0 & c_3L_3 \\ s_3 & c_3 & 0 & s_3L_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} c_4 & 0 & -s_4 & c_4L_4 \\ s_4 & 0 & c_4 & s_4L_4 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot$$

$$\begin{pmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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Newton Sums

To compute

$$R = \left\{ p_0(T) = 0, \ x_i = \frac{p_i(T)}{q(T)}, i = 1, 2, \ldots, n \right\}$$

we proceed along two steps:

1. apply Newton's formula for p_0;
2. compute in the quotient algebra to find the rest.

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the characteristic polynomial

To derive the characteristic polynomial p_0, the trace of L^i is $s_i = \displaystyle\sum_{\mathbf{z} \in V} \mu_{\mathbf{z}} L^i(\mathbf{z})$.

If $p_0(T) = \displaystyle\sum_{i=0}^{D} b_i T^{D-i}, \quad D = \#V, \quad b_i \in \mathbb{C}[\mathbf{x}]$,

then $\displaystyle\frac{p_0'(T)}{p_0(T)} = \sum_{\mathbf{z} \in V} \frac{\mu_{\mathbf{z}}}{T - L(\mathbf{z})} = \sum_{j \geq 0} \frac{\mathrm{trace}(L^j)}{T^{j+1}}$.

We have $p_0'(T) = \displaystyle\sum_{\ell=0}^{D-1} \sum_{j=0}^{D-\ell-1} \mathrm{trace}(L^j) b_\ell T^{D-\ell-j-1}$.

Apply Newton's formula: $(D-i) b_i = \displaystyle\sum_{j=0}^{i} \mathrm{trace}(L^j) b_{i-j}$.

This gives a linear system in the b_i's.

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separating form L

For any $v \in A$, A is the quotient algebra, we define

$$g_L(v,T) = \sum_{\mathbf{z} \in V} \mu_{\mathbf{z}} v(\mathbf{z}) \prod_{\substack{y \in V(p_0) \\ y \neq L(\mathbf{z})}} (T - y).$$

Observe

$$\frac{g_L(v, L(\mathbf{z}))}{g_L(1, L(\mathbf{z}))} = \frac{\displaystyle\sum_{\mathbf{z} \in V} \mu_{\mathbf{z}} v(\mathbf{z}) \prod_{\substack{y \in V(p_0) \\ y \neq L(\mathbf{z})}} (L(\mathbf{z}) - y)}{\displaystyle\sum_{\mathbf{z} \in V} \mu_{\mathbf{z}} \prod_{\substack{y \in V(p_0) \\ y \neq L(\mathbf{z})}} (L(\mathbf{z}) - y)} = v(\mathbf{z}).$$

Then we let v become a coordinate x_i ...

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As $\dfrac{g_L(v, L(\mathbf{z}))}{g_L(1, L(\mathbf{z}))} = v(\mathbf{z})$, we let $v = x_i$ and we have

$$x_i = \frac{g_L(x_i, T)}{g_L(1, T)}, \quad i = 1, 2, \ldots, n.$$

Note

$$g_L(1, T) = \frac{p_0'(T)}{\mathrm{GCD}(p_0'(T), p_0(T))}.$$

If all roots occur with multiplicity one,
the denominator $g_L(1, T)$ is just the derivative of $p_0(T)$.

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To compute $g_L(v, T)$, we define

$$\overline{p_0}(T) = \frac{p_0(T)}{\mathrm{GCD}(p_0'(T), p_0(T))},$$

and

$$g_L(v, T) = \sum_{j=0}^{d-1} \sum_{k=0}^{d-j-1} \mathrm{trace}(vL^j) a_i T^{d-j-k-1}, \quad \overline{p_0} = \sum_{i=0}^{d} a_i T^{d-i}.$$

Then we set $q(T) = g_L(1, T)$ and $p_i(T) = g_L(x_i, T)$ and obtain a representation for RUR.

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   computing the coordinates
4. Algorithm and Software
   pseudo code to compute a RUR, using Maple

pseudo code

Input: . $\mathcal{G}_>$, a Gröbner basis for an ideal I with term order $>$,
$\#V(I) = D < \infty$.

Output: a rational univariate representation for $V(\sqrt{I})$.

compute $\mathcal{N}_>$ the basis vector for the quotient ring;
let $D = \#\mathcal{N}_> = \#V(I)$, counted with multiplicities;
compute TrM and deduce $d = \#$distinct zeroes;
choose a separating element u as one of the u_i's;
compute for m from 1 to D: trace(u^m) and use u to form $p_0(T)$;
compute $\bar{p}_0$ for $\sqrt{I}$, if $\deg(\bar{p}_0) < d$ then choose another u;
for j from 1 to D
  for i from 0 to d
    compute trace($x_j u^i$) and deduce $g_u(x_j, T)$;
set $q(T) = g_u(1, T)$ and $p_i = g_u(x_i, T)$, $i = 1, 2, \ldots, n$.

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using Maple

In Maple, we compute a RUR as follows:

```
[> f := [2*x[1]^2 + 2*x[2]^2 + 2*x[3]^2 + x[4]^
        2*x[1]*x[2] + 2*x[2]*x[3] + 2*x[3]*x[4]
        2*x[1]*x[3] + x[3]^2 + 2*x[2]*x[4] - x[
        2*x[1] + 2*x[2] + 2*x[3] + x[4] - 1];
[> v := x[4],x[3],x[2],x[1];
[> Groebner[Basis](f,plex(v));
[> Groebner[RationalUnivariateRepresentation]
     (f,v,output=factored);
```

Summary + Exercises

RUR generalizes the Shape Lemma representation.

Exercises:

1. Construct an example of a solution set V in two variables and $\#V > 1$ where all except for one choice of the u_i's fail to be separating.
2. Use a lexicographic term order to compute a Gröbner basis for the system Katsura, for $n = 3$ (see lecture 3). How many decimal places does the largest coefficient in this basis have? Compare with the size of the coefficients in the lecture note.
3. Use Maple's `Groebner[RationalUnivariateRepresentation]` on the example of the previous exercise.

more exercises

4. Create a Maple worksheet to define the polynomial system for the elbow manipulator, for general choices of the position. Solve the system using the choices in the lecture note, either by Maple or by Sage.Singular.
5. Consider the system:

$$f(\mathbf{x}) = \begin{cases} 24x_1x_2 - x_1^2 - x_2^2 - x_1^2x_2^2 - 13 = 0 \\ 24x_2x_3 - x_2^2 - x_3^2 - x_2^2x_3^2 - 13 = 0 \\ 24x_3x_1 - x_3^2 - x_1^2 - x_3^2x_1^2 - 13 = 0. \end{cases}$$

   1. Solve the system to verify that no variable is separating.
   2. Find a separating element of the form as u_i an run through the steps of Algorithm to compute a RUR. Use a worksheet or a notebook to guide the computations.
   3. Use the builtin commands in Maple or Sage to compute a rational univariate representation. Compare the output with the outcome of the step-by-step execution of Algorithm to compute a RUR.

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one last exercise

6. Consider the following modification of the cyclic 5-roots problem:

$$
f(\mathbf{x}) = \left\{
\begin{array}{c}
x_1 + x_2 + x_3 + x_4 + x_5 = 0 \\
x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 = 0 \\
x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_1 + x_5x_1x_2 = 0 \\
x_2x_3x_4 + x_2x_3x_4x_5 + x_3x_4x_5x_1 + x_4x_5x_1x_2 + x_5x_1x_2x_3 = \\
x_1x_2x_3x_4x_5 - 1 = 0.
\end{array}
\right.
$$

where the monomial $x_1x_2x_3x_4$ in the original cyclic 5-roots system is replaced by $x_2x_3x_4$. Compute a Gröbner basis with the graded lexicographical order to determine the number of roots of this modified cyclic 5-roots system. Compute a rational univariate representation for this system. Compare the size of the coefficients between the lexicographical Gröbner basis (shape lemma) and the RUR.