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Prony's Method

interpolation for coefficients and exponents

Singularities of Stewart-Gough platforms

computing the determinant of the Jacobian matrix

Reformulating Prony's Method

deriving a generalized eigenvalue problem
the QZ method to solve $Ax = \lambda Bx$

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determining the size of the support

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MCS 563 Lecture 27
Analytic Symbolic Computation
Jan Verschelde, 28 March 2011

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interpolating exponentials

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Consider the exponential function $F(x) = c_1 e^{\mu_1 x} + c_2 e^{\mu_2 x}$ for *unknown* coefficients c_1 , c_2 , and exponents μ_1 , μ_2 .

To determine 4 unknowns, evaluate at $x = 0, 1, 2, 3$:

$$\left\{ \begin{array}{l} F(0) = c_1 + c_2 = c_1 + c_2 \\ F(1) = c_1 e^{\mu_1} + c_2 e^{\mu_2} = c_1 b_1 + c_2 b_2 \\ F(2) = c_1 e^{2\mu_1} + c_2 e^{2\mu_2} = c_1 b_1^2 + c_2 b_2^2 \\ F(3) = c_1 e^{3\mu_1} + c_2 e^{3\mu_2} = c_1 b_1^3 + c_2 b_2^3 \end{array} \right.$$

where $b_1 = e^{\mu_1}$ and $b_2 = e^{\mu_2}$.

The matrix of the linear system in c_1 and c_2 is a Vandermonde matrix.

an auxiliary polynomial

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We introduce an auxiliary polynomial

$$p(z) = (z - b_1)(z - b_2) = z^2 + \lambda_1 z + \lambda_0.$$

If we know the coefficients λ_1 and λ_0 of p , then b_1 and b_2 are found by computing the roots of p .

Once b_1 and b_2 are known, we find c_1 and c_2 by solving a linear system.

Because $p(b_1) = 0$ and $p(b_2) = 0$, we have

$$\begin{cases} c_1 p(b_1) + c_2 p(b_2) = 0 \\ c_1 b_1 p(b_1) + c_2 b_2 p(b_2) = 0. \end{cases}$$

a Hankel system

Recall $p(z) = z^2 + \lambda_1 z + \lambda_0$.

$$\begin{cases} c_1(b_1^2 + \lambda_1 b_1 + \lambda_0) + c_2(b_2^2 + \lambda_1 b_2 + \lambda_0) = 0 \\ c_1(b_1^3 + \lambda_1 b_1^2 + \lambda_0 b_1) + c_2(b_2^3 + \lambda_1 b_2^2 + \lambda_0 b_2) = 0. \end{cases}$$

Collecting terms in λ_0 and λ_1 gives

$$\begin{cases} \lambda_0(c_1 + c_2) + \lambda_1(c_1 b_1 + c_2 b_2) + c_1 b_1^2 + c_2 b_2^2 = 0 \\ \lambda_0(c_1 b_1 + c_2 b_2) + \lambda_1(c_1 b_1^2 + c_2 b_2^2) + c_1 b_1^3 + c_2 b_2^3 = 0. \end{cases}$$

As we recognize $F(0)$, $F(1)$, $F(2)$, and $F(3)$, we write the system in shortened form as

$$\begin{bmatrix} F(0) & F(1) \\ F(1) & F(2) \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \lambda_1 \end{bmatrix} = - \begin{bmatrix} F(2) \\ F(3) \end{bmatrix}.$$

The matrix is a Hankel matrix.

the method of Prony

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Prony's method performs three steps:

- 1 Solve the Hankel system to find the coefficients λ_1 and λ_0 of the auxiliary polynomial $p(z)$.
- 2 Solve $p(z) = 0$. The roots of p are b_1 and b_2 . So the exponents are $\mu_1 = \ln(b_1)$ and $\mu_2 = \ln(b_2)$.
- 3 Solve the first two interpolating condition for c_1 and c_2 . For known b_1 and b_2 , this system is linear.

This works of course for any number of exponentials.

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Stewart-Gough Platforms

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Think of the design of a flight simulator:

- A moving platform is supported above a stationary base by six legs.
- The i th leg connects to the base at the joint \mathbf{b}_i and to the platform at the joint at \mathbf{a}_i , $i = 1, 2, \dots, 6$.
- The leg length L_i is controlled by a linear actuator.

Understanding the singularities is important for designers.

The condition for singularity can be derived as follows:

- With quaternions we formulate the platform system.
- The determinant of the Jacobian matrix is a polynomial in several variables, which we compute explicitly.

deriving equations

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The platform is represented by

- its position $\mathbf{p} \in \mathbb{C}^3$; and
- the orientation by a quaternion $\mathbf{q} \in \mathbb{P}^3$.

In base coordinates, the vector \mathbf{v}_i connects base point \mathbf{b}_i to the corresponding platform point \mathbf{a}_i :

$$\mathbf{v}_i := -\mathbf{b}_i + \mathbf{p} + R(\mathbf{q})\mathbf{a}_i = -\mathbf{b}_i + \mathbf{p} + \mathbf{q}\mathbf{a}_i\mathbf{q}'/\mathbf{q}\mathbf{q}',$$

where $R(\mathbf{q})$ is the 3×3 rotation matrix giving the same rotation to a vector \mathbf{w} as the quaternion operation $\mathbf{q}\mathbf{w}\mathbf{q}'/\mathbf{q}\mathbf{q}'$.

relations on leg lengths

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The squared length of leg i is $L_i^2 = \mathbf{v}_i \cdot \mathbf{v}_i$, so

$$L_i \dot{L}_i = \mathbf{v}_i \cdot \dot{\mathbf{v}}_i = \mathbf{v}_i \cdot (\dot{\mathbf{p}} + \boldsymbol{\omega} \times (R\mathbf{a}_i)),$$

where “ \cdot ” is the vector inner product, \times is the vector cross product, and $\boldsymbol{\omega}$ is the angular velocity vector.

Rewriting $R = \hat{R}/Q$ with

$$Q := \mathbf{q}\mathbf{q}' = q_0^2 + q_1^2 + q_2^2 + q_3^2,$$

and substituting from the equation on \mathbf{v}_i
we transform the relation on the leg lengths into

$$QL_i \dot{L}_i = (Q(\mathbf{p} - \mathbf{b}_i) + \hat{R}\mathbf{a}_i) \cdot \dot{\mathbf{p}} + ((\hat{R}\mathbf{a}_i) \times (\mathbf{p} - \mathbf{b}_i)) \cdot \boldsymbol{\omega}.$$

the Jacobian matrix

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Letting \mathbf{J} be the matrix whose i -th column is

$$\mathbf{J}_i = \begin{bmatrix} Q(\mathbf{p} - \mathbf{b}_i) + \widehat{R}\mathbf{a}_i \\ (\widehat{R}\mathbf{a}_i) \times (\mathbf{p} - \mathbf{b}_i) \end{bmatrix}, \quad i = 1, 2, \dots, 6,$$

the singularity condition is $0 = \mathbf{J}\mathbf{w}$ for some $\mathbf{w} \neq 0$,
or equivalently, $\det(\mathbf{J}) = 0$.

The expanded $\det(\mathbf{J})$ defines the polynomial we will factor.

Although we start out with quadratic polynomials, $\det(\mathbf{J})$ is a
homogeneous polynomial of degree 1728 in 42 variables.

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Some case studies:

1 General platform, fixed position:

In this case, $\det(\mathbf{J})$ is a homogeneous polynomial of degree 12 with 910 terms in $\mathbf{q} = \{q_0, q_1, q_2, q_3\}$.

The factorization is $\det(\mathbf{J}) = F_1(\mathbf{q})Q^3$,
where $F_1(\mathbf{q})$ is a sextic.

The factor of $Q^3 = 0$ is not of physical significance.

2 Planar base and platform, fixed position:

$\det(J)$ is still homogeneous of degree 12,
and still factors in two:

one irreducible single factor of degree six,
and the quadratic factor having multiplicity three.

third case study

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- 3 Planar base and platform, parallel planes:
In this case, $\det(\mathbf{J})$ becomes cubic in \mathbf{p} ,
homogeneous of degree 12 in (q_0, q_3) ,
and degree 15 in (\mathbf{p}, \mathbf{q}) together.

This polynomial is much sparser: only 24 terms.

The computed factorization is

$$\det(\mathbf{J}) = ap_3^3(q_0 + bq_3)(q_0 + cq_3)(q_0 + iq_3)^5(q_0 - iq_3)^5,$$

where a, b, c are constants that depend on the choice
of $\mathbf{a}_i, \mathbf{b}_i$.

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numerical problems

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The three steps in Prony's algorithm all suffer numerical problems:

- 1 a Hankel matrix might be very ill-conditioned;
- 2 the roots are often very sensitive to small changes in the coefficients;
- 3 Vandermonde matrices are also typically ill-conditioned.

Therefore, we reformulate the problem.

matrix decomposition

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As in our two exponentials as before:

$$\begin{bmatrix} 1 & 1 \\ b_1 & b_2 \end{bmatrix} \underbrace{\begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}} \begin{bmatrix} 1 & b_1 \\ 1 & b_2 \end{bmatrix} = \underbrace{\begin{bmatrix} F(0) & F(1) \\ F(1) & F(2) \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 1 \\ b_1 & b_2 \end{bmatrix} \underbrace{\begin{bmatrix} c_1 & c_1 b_1 \\ c_2 & c_2 b_2 \end{bmatrix}} = \underbrace{\begin{bmatrix} c_1 + c_2 & c_1 b_1 + c_2 b_2 \\ c_1 b_1 + c_2 b_2 & c_1 b_1^2 + c_2 b_2^2 \end{bmatrix}},$$

abbreviated as $VDV^T = H_0$.

matrix multiplications

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Multiply the diagonal matrix D with another diagonal matrix that has b_1 and b_2 on its diagonal:

$$\begin{bmatrix} 1 & 1 \\ b_1 & b_2 \end{bmatrix} \underbrace{\begin{bmatrix} c_1 b_1 & 0 \\ 0 & c_2 b_2 \end{bmatrix}} \begin{bmatrix} 1 & b_1 \\ 1 & b_2 \end{bmatrix} = \underbrace{\begin{bmatrix} F(1) & F(2) \\ F(2) & F(3) \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 1 \\ b_1 & b_2 \end{bmatrix} \underbrace{\begin{bmatrix} c_1 b_1 & c_1 b_1^2 \\ c_2 b_2 & c_2 b_2^2 \end{bmatrix}} = \underbrace{\begin{bmatrix} c_1 b_1 + c_2 b_2 & c_1 b_1^2 + c_2 b_2^2 \\ c_1 b_1^2 + c_2 b_2^2 & c_1 b_1^3 + c_2 b_2^3 \end{bmatrix}},$$

abbreviated as $VDBV^T = H_1$.

generalized eigenvalue problem

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The two Hankel matrices H_0 and H_1 are diagonalized simultaneously:

$$V^{-1}H_0V^{-T} = D$$

$$V^{-1}H_1V^{-T} = DB.$$

Therefore, we obtained the generalized eigenvalue problem

$$(H_1 - \lambda H_0)\mathbf{v} = \mathbf{0}.$$

This reformulation of Prony's method as a generalized eigenvalue problem avoids the computation of the coefficients of the auxiliary polynomial and the corresponding root finding problem.

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generalized eigenvalue problem

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A generalized eigenvalue problem is

$$A\mathbf{x} = \lambda B\mathbf{x}, \quad A, B \in \mathbb{C}^{n \times n},$$

where λ is a scalar and \mathbf{x} an n -vector.

We look for eigenvalues $\lambda \in \mathbb{C}$ so that $(A - \lambda B)\mathbf{x} = \mathbf{0}$ admits an eigenvector $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x} \neq \mathbf{0}$.

If B has an inverse B^{-1} , then we could solve $B^{-1}A\mathbf{x} = \lambda\mathbf{x}$.

An ill conditioned matrix B prevents accurate computation of well conditioned eigenvalue-eigenvector pairs.

the Schur decomposition

We first compute the Schur decomposition of A and B :

$$\begin{cases} Q^H A Z = T, & Q^H Q = I, Z^H Z = I, \\ Q^H B Z = S, & T \text{ and } S \text{ are upper triangular.} \end{cases}$$

I represents the identity matrix.

By \cdot^H we denote the conjugate transposition: $(a_{ij})^H = \bar{a}_{ji}$.

Applying the transformations to $A\mathbf{x} = \lambda B\mathbf{x}$ gives

$$Q^H(A - \lambda B)Z = Q^H A Z - \lambda Q^H B Z = T - \lambda S,$$

reducing $A\mathbf{x} = \lambda B\mathbf{x}$ to $T\mathbf{x} = \lambda S\mathbf{x}$.

Givens rotations introduce zeroes:

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} z \\ 0 \end{bmatrix}, \quad z = \sqrt{x^2 + y^2}, \quad c = \frac{x}{z}, \quad s = -\frac{y}{z},$$

where $c = \cos(\theta)$ and $s = \sin(\theta)$ for the rotation angle θ .

Hessenberg matrices

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The first step to obtain a Schur decomposition is to compute a Hessenberg-triangular reduction.

A matrix $A = (a_{ij})$ is upper Hessenberg if $a_{ij} = 0$ for all $i > j + 1$.

An upper Hessenberg matrix is almost upper triangular, except of the elements just below the diagonal.

Hessenberg-triangular reduction of (A, B)

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Algorithm Hessenberg-Triangular Reduction of (A, B)

Input: $A, B \in \mathbb{C}^{n \times n}$.

Output: $Q, Z, T, S \in \mathbb{C}^{n \times n}$: $Q^H Q = I, Z^H Z = I,$

$Q^H A Z = T, T$ is upper Hessenberg,

$Q^H B Z = S, S$ is upper triangular.

$[Q^H, B] := QR(B);$

$A := Q^H A; Z := I;$

for j from 1 to $n - 1$ do

 for i from $n + 1 - j$ to $j + 2$ do

$A := Q_{ij}^H A; B := Q_{ij}^H B; Q := Q_{ij}^H Q;$

$A := AZ_{ij}; B := BZ_{ij}; Z := ZZ_{ij};$

return (Q, Z, A, B) .

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The problem of sparse interpolation is to determine the number of monomials in the polynomial

$$f(\mathbf{x}) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad m = \#A.$$

Given is a blackbox function to evaluate f at roots of unity ω_k yielding values $v_k = f(\omega_k)$, $k = 0, 1, \dots, 2m - 1$.

The problem is to determine m .

Hadamard polynomials

For $m = 3$, $p(x) = x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0$ where

$$\begin{bmatrix} v_0 & v_1 & v_2 \\ v_1 & v_2 & v_3 \\ v_2 & v_3 & v_4 \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = - \begin{bmatrix} v_3 \\ v_4 \\ v_5 \end{bmatrix}.$$

Moving the righthandside of the above equation to the left and adding $\lambda_0 + \lambda_1 x + \lambda_2 x^2 + x^3 = 0$ gives

$$\begin{bmatrix} v_0 & v_1 & v_2 & v_3 \\ v_1 & v_2 & v_3 & v_4 \\ v_2 & v_3 & v_4 & v_5 \\ 1 & x & x^2 & x^3 \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The determinant of this matrix equation is the polynomial denoted by $H_3^{(0)}(x)$, where its leading coefficient is $\det H_3^{(0)}$. Observe: if $m = 2$, then $\det H_3^{(0)} = 0$.

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A k -by- k Hankel matrix $H_k^{(j)}$ is defined by

$$H_k^{(j)} = \begin{bmatrix} v_j & v_{j+1} & \cdots & v_{j+k-1} \\ v_{j+1} & v_{j+2} & \cdots & v_{j+k} \\ \vdots & \vdots & \ddots & \vdots \\ v_{j+k-1} & v_{j+k} & \cdots & v_{j+2k-2} \end{bmatrix}.$$

Neighboring Hankel determinants satisfy

$$\left(\det H_k^{(j)} \right)^2 = \det H_k^{(j-1)} \det H_k^{(j+1)} - \det H_{k+1}^{(j-1)} \det H_{k-1}^{(j+1)}.$$

The qd algorithm bypasses the computation of Hankel determinants.

the qd algorithm

The initialization of the qd algorithm goes as

$$e_0^{(j)} = 0, \quad j = 0, 1, \dots, \quad q_1^{(j)} = \frac{v_{j+1}}{v_j}, \quad j = 1, 2, \dots$$

and for $k = 1, 2, \dots$:

$$q_k^{(0)} = \frac{\det H_{k-1}^{(0)} \det H_k^{(1)}}{\det H_k^{(0)} \det H_{k-1}^{(1)}}, \quad e_k^{(0)} = \frac{\det H_{k+1}^{(0)} \det H_{k-1}^{(1)}}{\det H_k^{(0)} \det H_k^{(1)}}.$$

Then the progressive qd algorithm uses the formulas

$$q_k^{(j+1)} = e_{k-1}^{(j+1)} - e_k^{(j)} + q_k^{(j)}, \quad e_k^{(j+1)} = \left(\frac{q_{k+1}^{(j)}}{q_k^{(j+1)}} \right) e_k^{(j)},$$

for $k = 1, 2, \dots$ and $j = 0, 1, \dots$

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The q and the d of the qd algorithm stand for quotient and difference.

The qd algorithm computes both the roots z_k of the polynomial p and its degree m .

The key properties are:

- 1 $\lim_{j \rightarrow \infty} q_k^{(j)} = z_k$; and
- 2 $e_m^{(j)} = 0$ for all $j = 1, 2, \dots$

Summary + Exercises

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A numerically stable version of Prony's method uses the qd algorithm and the QZ method for sparse interpolation.

Exercises:

- 1 Suppose a polynomial in one variable is given as a blackbox function: evaluation is cheap, but we do not know its algebraic representation. Explain by example(s) how divided differences and Newton interpolation lead to the degree of the polynomial.

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- 3 Generate Vandermonde matrices of increasing dimension (using MATLAB or Octave) and compute their condition number.

Take interpolation points in $[-1, +1]$ and compare the condition numbers for (1) equidistant interpolation points; (2) randomly chosen points, from a uniform distribution; and (3) for interpolation points as the roots of the Chebyshev polynomials.

- 4 Verify $H\mathbf{c} = \pm \|\mathbf{c}\|_2 \mathbf{e}_1$, for $\mathbf{e}_1 = [1 \ 0 \ 0 \ \cdots \ 0]^T$,
 $\mathbf{u} = \mathbf{c} \pm \|\mathbf{c}\|_2 \mathbf{e}_1$, and $H = \left(I - 2 \frac{\mathbf{u}\mathbf{u}^T}{\mathbf{u}^T \mathbf{u}} \right)$.

and more exercises

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- 5 Use the QZ algorithm as implemented in MATLAB or Octave to verify an instance of $VDV^T = H_0$.
- 6 Consider $f(x, y) = x^5y + 0.1xy^{13} - 0.5xy + 2.2x^4y^4$. Describe how to recover the exponents and coefficients, knowing that f has four terms.