Tropical Algebraic Geometry

MCS 563 Lecture 36
Analytic Symbolic Computation
Jan Verschelde, 18 April 2011

Tropical Algebraic Geometry

logarithms of algebraic sets

Denoting $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, we consider the application of

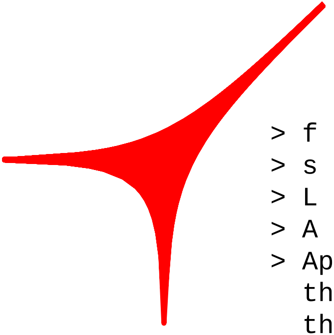
$$\begin{array}{rcccc} \log & : & \mathbb{C}^* \times \mathbb{C}^* & \rightarrow & \mathbb{R} \times \mathbb{R} \\ & & (x, y) & \mapsto & (\log(|x|), \log(|y|)) \end{array}$$

to a variety.

Introduced by Gelfand, Kapranov, and Zelevinsky, $\log(V)$ for a variety V is called *the amoeba of a variety*.

plot for a line

For a line, we use polar coordinates to plot the amoeba:
$f := \frac{1}{2}x + \frac{1}{5}y - 1 = 0$, $A := \left[\ln\left(\left|re^{I\theta}\right|\right), \ln\left(\left|\frac{5}{2}re^{I\theta} - 5\right|\right)\right]$.



```
> f := 1/2*x + 1/5*y - 1:
> s := solve(f,y):
> L := map(log,map(abs,[x,s])):
> A := subs(x=r*exp(I*theta),L);
> Ap := seq(plot([op(subs(
  theta=k*Pi/200,A)),r=-100..100],
  thickness=6),k=0..99):
> plots[display](Ap,axes=none);
```

The amoeba of a linear polynomial with Maple.

compactifying the amoeba

We compactify the amoeba of $f^{-1}(0)$:

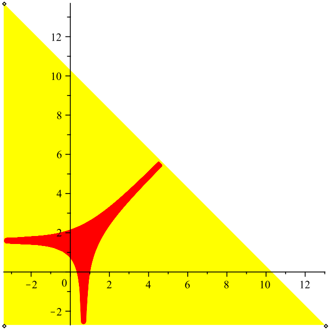
- Take lines perpendicular to the tentacles.
- As each line cuts the plane in half, keep those halves of the plane where the amoeba lives.

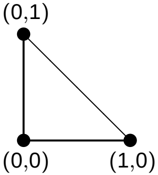
The intersection of all half planes defines a polygon.

The resulting polygon is the Newton polygon of f.

amoeba and Newton polygon

The edges of the Newton polygon are perpendicular to the tentacles of the amoeba.





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normal cones and fan

The *inner product* is

$$\langle \cdot, \cdot \rangle : \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{Z} : ((i,j),(u,v)) \mapsto iu + jv.$$

Let P be the Newton polygon of the polynomial f.

The *normal cone to a vertex* $\mathbf{p}$ *of* P is

$$\{ \mathbf{v} \in \mathbb{R}^2 \setminus \{0\} \mid \langle \mathbf{p}, \mathbf{v} \rangle = \min_{\mathbf{q} \in P} \langle \mathbf{q}, \mathbf{v} \rangle \}.$$

The *normal cone to an edge spanned by* $\mathbf{p}_1$ *and* $\mathbf{p}_2$ is

$$\{ \mathbf{v} \in \mathbb{R}^2 \setminus \{0\} \mid \langle \mathbf{p}_1, \mathbf{v} \rangle = \langle \mathbf{p}_2, \mathbf{v} \rangle = \min_{\mathbf{q} \in P} \langle \mathbf{q}, \mathbf{v} \rangle \}.$$

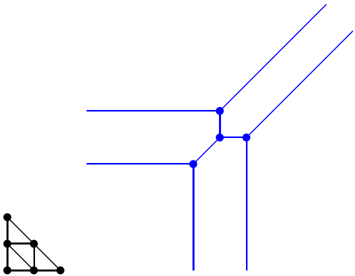
The *normal fan* of P is the collection of all normal cones to vertices and edges of P.

Given f, *a tropicalization of* f, denoted by $\mathrm{Trop}(f)$, is a finite collection of inner normals (u,v) (normalized as $\gcd(u,v) = 1$) to the edges of the Newton polygon P of f.

a quadratic polynomial

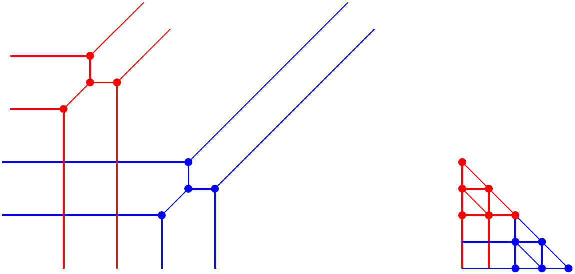
Consider a regular triangulation of the Newton polygon of a general quadratic polynomial.

Take the normals to the facets of the *upper* hull:



intersecting two quadrics

Consider two general quadratic polynomials:



We see a tropical version of the theorem of Bézout.

This tropical version is an interpretation of the computation of the mixed cells in a regular mixed-cell configuration via the common refinement of the normal fans to their edges.

a dictionary

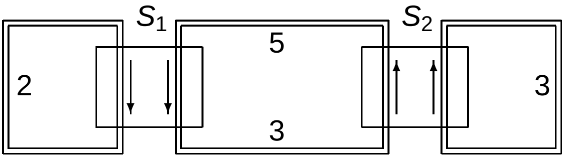
An analogy between classical algebraic geometry and tropical algebraic geometry:

classical	$\leftrightarrow$	tropical
$\mathbb{C}, +, \cdot$		$\mathbb{R}$, min, $+$
polynomial		piecewise linear function
zero set		singular locus
hypersurface		normal fan
toric variety		ordinary linear variety
algebraic variety		balanced polyhedral fan

Tropical Algebraic Geometry

a railway network

Consider two stations: S_1 and S_2; three circuits: one inner between the two stations and two outer circuits, serving the suburbs. Four trains travel back and forth.



Numbers along the tracks are fixed travel times.
A good timetable obey the rules:

1. the frequency of trains should be as high as possible;
2. the frequency of trains is the same along all tracks;
3. trains wait to allow the changeover of passengers;
4. the trains depart at a station as soon as allowed.

inequalities

Denote by x_1 and x_2 the common departure times of the trains respectively at S_1 and S_2.

The initial departure times are $x_1(0)$ and $x_2(0)$ and the kth departure times are given by $x_1(k-1)$ and $x_2(k-1)$.

The rules above translate into the inequalities:

$$x_1(k+1) = \max(x_1(k)+2, x_2(k)+5)$$
$$x_2(k+1) = \max(x_1(k)+3, x_2(k)+3)$$

In the max-plus algebra,
replacing max by $\oplus$ and addition by $\otimes$, we write

$$x_1(k+1) = (x_1(k) \otimes 2) \oplus (x_2(k) \otimes 5)$$
$$x_2(k+1) = (x_1(k) \otimes 3) \oplus (x_2(k) \otimes 3)$$

tropical linear algebra

In matrix-vector form:

$$\mathbf{x}(k+1) = A \otimes \mathbf{x}(k), \quad \mathbf{x}(k) = \left(\begin{array}{c} x_1(k) \\ x_2(k) \end{array} \right), \quad A = \left(\begin{array}{cc} 2 & 5 \\ 3 & 3 \end{array} \right).$$

Eigenvalues λ and eigenvectors $\mathbf{v}$ of a matrix A are defined in the max-plus algebra as

$$A \otimes \mathbf{v} = \lambda \otimes \mathbf{v},$$

where not all components of $\mathbf{v}$ are equal to $-\infty$.

tropical eigenvalues

Eigenvectors are (similar to conventional linear algebra) not uniquely determined.

If the same constant is added to all components of an eigenvector, then we get again an eigenvector.

It is typical to normalize an eigenvector so its first component is zero.

For the solution of the system above, we write

$$\mathbf{x}(1) = A \otimes \mathbf{x}(0) = \lambda \otimes \mathbf{x}(0)$$

and in general

$$\mathbf{x}(k) = \lambda^{\otimes k} \otimes \mathbf{x}(0), \quad k = 1, 2, \ldots$$

Tropical Algebraic Geometry

formal power series

The discrete valuation ring of formal power series in $t^{1/N}$ is

$$R_N = \mathbb{K}[[t^{1/N}]] = \left\{ \sum_{\alpha=0}^{\infty} c_\alpha t^{\alpha/N} \mid c_\alpha \in \mathbb{K} \right\}, \text{ for } N > 0,$$

with discrete valuation on $s \in R_N$:

$$\mathrm{val}(s) = \mathrm{ord}_t(s) = \min \left\{ \frac{\alpha}{N} \mid c_\alpha \neq 0 \right\} \in \frac{1}{N}\mathbb{Z} \cup \{\infty\}.$$

If N divides M, then R_N is a subring of R_M.
Denote the quotient field of R_N as L_N. The direct limit of L_N gives the field of fractional power (or Puiseux) series

$$L = \mathbb{K}\{\{t\}\} = \varinjlim L_N = \bigcup_{N>0} L_N.$$

Puiseux: if $\mathbb{K}$ is algebraically closed, then so is $\mathbb{K}\{\{t\}\}$.

tropical varieties

For an ideal J in the polynomial ring $\mathbb{K}\{\{t\}\}[\mathbf{x}]$, there are two ways to define the tropical variety of J, denoted by $\mathrm{Trop}(J)$:

1. using t-initial ideals:

$$\mathrm{Trop}(J) = \{ \, \omega \in \mathbb{R}^n \mid t - in_\omega(J) \text{ is monomial free } \} \, ,$$

   where $t - in_\omega$
   uses the weight $(-1, \omega)$ on monomials in $\mathbb{K}[t, \mathbf{x}]$;

or

2. $\mathrm{Trop}(J)$ is the image of $-\mathrm{val}(p)$
   for all $p \in V(J)$ of J in $\mathbb{K}\{\{t\}\}^n$.

the main theorem

The main theorem of tropical algebraic geometry states that both ways to defined tropical varieties are equivalent, or:

Theorem (the fundamental theorem of tropical algebraic geometry)

$$\omega \in \mathrm{Trop}(J) \cap \mathbb{Q}^n \Leftrightarrow \exists p \in V(J) : -\mathrm{val}(p) = \omega \in \mathbb{Q}^n.$$

We can rephrase the theorem as follows:

every rational vector in the tropical variety corresponds to the leading powers of a Puiseux series converging to a point in the variety.

t-initial ideals

A t-initial ideal of an ideal J is ***not*** generated by taking the t-initial forms of the given generators of J.

Consider for example $J = \langle tx + y, x + t \rangle$ as an ideal in $\mathbb{K}\{\{t\}\}[x, y]$.

We have that $y - t^2 \in J$. Let $\omega = (1, -1)$, so $y = t - in_\omega(y - t^2)$ belongs to the t-initial ideal.

However $\langle t - in_\omega(tx + y), t - in_\omega(x + t) \rangle = \langle x \rangle$ and $y \notin \langle x \rangle$.

To compute t-initial ideals, standard bases are used.

prime ideals

For pure dimensional prime ideals, the situation is better:

Theorem

Let I and ideal in $\mathbb{C}\{\{t\}\}[\mathbf{x}]$. If $\sqrt{I}$ is prime of dimension d, then $\mathrm{Trop}(I)$ is a pure d-dimensional polyhedral fan.

A finite intersection of tropical hypersurufaces is *a tropical prevariety*. Every ideal I has a fine generating set $\{f_1, f_2, \ldots, f_r\}$ such that

$$\mathrm{Trop}(I) = \mathrm{Trop}(f_1) \cap \mathrm{Trop}(f_2) \cap \cdots \cap \mathrm{Trop}(f_r).$$

This set of generators is *a tropical basis* for the ideal.
Every tropical variety is a tropical prevariety, but the converse does not hold.
An algorithm to compute the tropical prevariety uses the common refinement of the polyhedral fans of hypersurfaces.

Summary + Exercises

Tropical algebraic geometry provides a framework to apply polyhedral methods to solve polynomial systems.

Exercises:

1. Consider the amoeba of the product of two linear equations. How many tentacles to you see in each direction? For which choices of the coefficients do you see holes in the amoeba?
2. For ideals I and J show that $I \subset J \Rightarrow \mathrm{Trop}(I) \supset \mathrm{Trop}(J)$; $\mathrm{Trop}(I \cap J) = \mathrm{Trop}(I) \cup \mathrm{Trop}(J)$; and $\mathrm{Trop}(I + J) \subset \mathrm{Trop}(I) \cap \mathrm{Trop}(J)$.