MCS 563 Project One:

Exploring the Central Theorem in Numerical Polynomial Algebra

In the lectures we stated a simple version of the central theorem in Numerical Polynomial Algebra often, but never got to the full version. The main goal of this project is to understand the complete version of this central theorem and to gain computational experience, using Maple on an example:

$$P(x_1, x_2, x_3) = \begin{cases} x_1 + x_2 + x_3 - 1 = 0 \\ \frac{1}{5}x_1^3 + \frac{1}{2}x_2^2 - x_3 + \frac{1}{2}x_3^2 + \frac{1}{2} = 0 \\ x_1 + x_2 + \frac{1}{2}x_3^2 - \frac{1}{2} = 0 \end{cases}$$

While in this course, a Groebner basis is treated as just one kind of the many possible bases, they are the engine behind many polynomial computations in computer algebra systems like Maple. In this project, we will take the computation of a Groebner basis for granted.

0. Assignment Zero: Solve the System

Use Maple's **solve** command to solve the system.
Compare the output of this command with what **Groebner[gsolve]** returns.
Interpret the results by computing a Groebner basis for the ideal generated by P using a *pure lexicographic term order*. Simply by looking at the form of this Groebner basis, how many solutions do you expect?

1. Assignment One: Compute a Normal Set

Compute a Groebner basis using a *total degree term order*. Use this Groebner basis to compute a normal set, with the command **Groebner[SetBasis]**.

Visualize the normal set. Looking at the normal set, what can you say about the number of solutions?

2. Assignment Two: Eigenvalues of Multiplication Matrices

Using the normal set and the rest of the output computed in Assignment One to compute the multiplication matrices M_{x_1}, M_{x_2}, and M_{x_3}, using the command **Groebner[MulMatrix]**.

1. Verify that the multiplication matrices commute, e.g.: $M_{x_1}M_{x_2} = M_{x_2}M_{x_1}$.
2. Compute the eigenvalues of M_{x_1}, M_{x_2}, and M_{x_3} with the command **LinearAlgebra[Eigenvalues]**. Interpret the output by writing down all solutions of the system.

3. Assignment Three: Eigenvectors of Multiplication Matrices

Continuing with the multiplication matrices computed in Assignment Two, answer the following questions:

1. Look at the output of **LinearAlgebra[Eigenvectors]**(M_{x_1}). Why is this output not so useful?
2. Use **LinearAlgebra[JordanForm]** to compute the Jordan canonical form J of M_{x_1}.
   With **option='Q'**, compute also the invariant subspaces Q associated with the eigenvalues of M_{x_1}.
   Verify that $J = Q^{-1}M_{x_1}Q$.

At this stage, we have an example of Theorem 2.27 (Central Theorem) of page 52. Translate the abstract concepts of Theorem 2.27 into the specific objects in this example, i.e.: for every mathematical symbol in Theorem 2.27, give the corresponding result computed with Maple.

4. The deadline is Wednesday 22 September 2004 at 1PM

You are allowed to work in pairs (i.e.: groups of two). If you are not so familiar with Maple, team up with some one who knows Maple well. Every pair should hand in one report with the two names of its authors (not two identical reports).

The report should be structured along the assignments, i.e.: the first section contains the answers of Assignment Zero, followed by the answers of Assignment One, etc.

You may use Maple output in your answers, but the complete print out of your Maple worksheet must remain an *appendix* to your report.

Feel free to come to my office for help.