Midterm Exam MCS 563 Jan Verschelde Wednesday 28 February 2007

This is a take home exam. Your answers to the questions below are due on Monday 5 March, at 9AM (start of our regular class meeting). Every question counts for 20 points.

1. Consider the system $f(\mathbf{x}) = \begin{cases} x_1^2 + x_2^3 + 1 = 0 \\ x_1 x_2^2 - 1 = 0. \end{cases}$

   (a) Apply the theorem of Bézout to compute a bound on the number of solutions of $f(\mathbf{x}) = \mathbf{0}$.

   (b) Compute the 2-homogeneous Bézout number and give the structure of the corresponding linear-product start system.

   (c) Use elimination to reduce the system to one polynomial in one variable to compute the number of solutions.

2. Consider the polynomial $p = x^3 + ax^2 - x + b$, with parameters a and b.

   (a) Derive a condition on the parameters a and b for which p has singular solutions.

   (b) For which values of a and b are all three roots of p real?

3. Consider the system $f(x, y) = \begin{cases} x^2 + y - 3 = 0 \\ x + 0.125y^2 - 1.5 = 0. \end{cases}$

   The program `phc -b` returns four solutions for this system, see the back of this sheet.

   (a) Interpret the output. Are the solutions well conditioned?

   (b) If we would add to the coefficients of the system a small term of size ϵ, then δ_k denotes the size of the change to the kth solution. For every solution k, estimate the size of δ_k.

4. Consider the system $f(\mathbf{x}) = \begin{cases} x_1^3 + x_1^2 - x_1 x_2 = 0 \\ x_1 - x_2 + 1 = 0. \end{cases}$

   (a) Bring this system in the form of the shape lemma.

   (b) Compute the multiplication matrix M_{x_1}, using $\{1, x_1, x_1^2\}$ as basis for the quotient ring.

   (c) What are the eigenvalues of M_{x_1}? Compare the eigenvalues with the solutions of the system.

5. Consider the system $f(\mathbf{x}) = \begin{cases} x_1^2 + x_2^3 + 1 = 0 \\ x_1 x_2^2 - 1 = 0. \end{cases}$

   (a) Draw the sum of the two Newton polygons of f. Use the drawing to construct a regular mixed-cell configuration to compute the mixed area of the Newton polygons of f.

   (b) Show that the mixed volume for this system is sharp, i.e.: for any nonzero choice of the coefficients, the system always has exactly as many solutions as the mixed volume.

```
2
 x**2+y-3;
 x+0.125*y**2-1.5;

THE SOLUTIONS :
4 2
===========================================================
solution 1 :
t :  1.00000000000000E+00   0.00000000000000E+00
m : 0
the solution for t :
 x :  9.99997059717117E-01   5.91574800619624E-06
 y :  2.00000588058572E+00  -1.18314656064162E-05
== err :  1.045E-05 = rco :  1.437E-11 = res :  1.263E-11 ==
solution 2 :
t :  1.00000000000000E+00   0.00000000000000E+00
m : 0
the solution for t :
 x :  1.00000781255449E+00   8.76354714457772E-07
 y :  1.99998437483211E+00  -1.75271487765531E-06
== err :  1.222E-05 = rco :  1.767E-11 = res :  1.126E-11 ==
solution 3 :
t :  1.00000000000000E+00   0.00000000000000E+00
m : 1
the solution for t :
 x : -3.00000000000000E+00   0.00000000000000E+00
 y : -6.00000000000000E+00   0.00000000000000E+00
== err :  3.400E-16 = rco :  3.163E-01 = res :  4.441E-16 ==
solution 4 :
t :  1.00000000000000E+00   0.00000000000000E+00
m : 0
the solution for t :
 x :  1.00000675073431E+00   2.65806658263181E-07
 y :  1.99998649849241E+00  -5.31615198867799E-07
== err :  8.726E-06 = rco :  1.157E-11 = res :  9.655E-12 ==
```

The real and imaginary part of the coordinates of each solution for x and y are listed respectively following the `x :` and `y :` labels. The diagnostics mean the following:

`err` : magnitude of the correction term of Newton's method,
i.e.: $||\Delta(x,y)||$, where $\Delta(x,y) = -J_f^{-1}(x,y)f(x,y)$, J_f is the Jacobian matrix of f.

`rco` : estimate for the inverse of the condition number of the Jacobian matrix
at the computed solution.

`res` : magnitude of the residual, i.e.: $||f(x,y)||$.