

Review and Summary of the first part of MCS 563

Our object of study is a system $f(\mathbf{x}) = \mathbf{0}$ of N polynomials $f = (f_1, f_2, \dots, f_N)$, with complex coefficients: $f_i \in \mathbb{C}[\mathbf{x}]$, $i = 1, 2, \dots, N$, in n variables: $\mathbf{x} = (x_1, x_2, \dots, x_n)$. We are interested in algorithms to compute and describe the solution set $V = f^{-1}(\mathbf{0})$.

The topics are summarized in the titles of the lectures:

1/17	Lec 1	introduction (Newton and Bézout)
1/19	Lec 2	elimination methods (resultants)
1/22	Lec 3	homotopies and predictor-corrector methods
1/24	Lec 4	rewriting polynomials (division algorithm)
1/26	Lec 5	multihomogenization (linear-product)
2/05	Lec 6	Gröbner bases (Buchberger's algorithm)
2/06	Lec 7	linear product structures (symmetry)
2/07	Lec 8	quotient rings (shape lemma)
2/09	Lec 9	coefficient-parameter homotopies (cheater)
2/12	Lec 10	Gröbner basis conversion (FGLM)
2/13	Lec 11	Condition and Scaling
2/14	Lec 12	Rational Univariate Representation
2/16	Lec 13	Real Homotopies (turning points)
2/19	Lec 14	Kronecker Parametrization (RUR again)
2/20	Lec 15	Kushnirenko's theorem (Newton polytopes)
2/21	Lec 16	mixed volumes (Cayley trick)
2/23	Lec 17	Bernshtein's second theorem
2/26	Lec 18	Polyhedral Homotopies

We can classify the topics in three categories:

Numerical Homotopy Continuation: Starting in the first lecture with Newton's method, homotopies were introduced in lecture three. Every odd numbered lecture from 3 till 13 considered polynomial systems from a numerical perspective.

Symbolic Rewriting and Elimination: In lecture 2 we used resultants to introduce the main theorem of elimination theory. In every even numbered lecture from 2 till 14, we looked at polynomial systems from a symbolic perspective, using term rewriting techniques as solution methods.

Polyhedral Methods: Newton polytopes were introduced in lecture 15 as a way to model the sparse structure of a polynomial system. In the last four lectures, we stated the theorems of Kushnirenko and Bernshtein, ending with polyhedral homotopies.

Continuing the classification of materials in a tripartite fashion, we distinguish concepts, theorems, and algorithms. The exercises usually provide more examples of definitions encountered in the lectures and give opportunities to explore the algorithms.

A typical question we ask when we see a polynomial system is to bound the number of isolated solutions. We have seen two generalizations of Bézout's theorem and used mixed volumes to compute a root count. With every root count we associate a start system to be used in a homotopy.

An alternative look at a polynomial system applies elimination, either explicitly via resultants or the shape lemma, or more implicitly via eigenvalue problems and rational univariate parameterization. The main tool here is a Gröbner basis which turns the division algorithm into a normal form algorithm allowing computations in the quotient ring.

In addition to studying algorithms to solve polynomial systems, an essential aspect of the course has been the connection with practical applications. The titles below gives an overview of the application areas in which polynomial systems arise:

1/17	Lec 1	planar robot arm
1/19	Lec 2	molecular configurations
1/22	Lec 3	problem from magnetics
1/24	Lec 4	automatic geometric theorem proving
1/26	Lec 5	Nash equilibria
2/05	Lec 6	Wavelet design
2/06	Lec 7	Neural networks
2/07	Lec 8	design of experiments
2/09	Lec 9	Stewart-Gough platforms
2/12	Lec 10	cyclic n-roots
2/13	Lec 11	chemical equilibria
2/14	Lec 12	the elbow manipulator
2/16	Lec 13	Newtonian n-body problem
2/19	Lec 14	camera motion
2/20	Lec 15	design of 4-bar mechanisms
2/21	Lec 16	RPS serial chains
2/23	Lec 17	mass action kinetics
2/26	Lec 18	Runge-Kutta formulas

The Stewart-Gough platforms were explored in the first computer project.

The best way to prepare for the midterm exam is to solve the exercises with each lecture. In any case, homework will be collected on Monday 5 March.

The midterm exam is a take home exam and contains some typical exercises regarding the core material of the course. Its due date is also Monday 5 March, at 9AM (start of the class).