Approximate Factorization

1. The Ruppert Matrix
   - a criterion for irreducibility
2. an Open Problem
   - polynomial time in symbolic-numeric computing
3. The Kernel of the Ruppert Matrix
   - relation between rank and greatest common divisor
4. SVD and Approximate GCD
   - symbolic-numeric algorithm for approximate factorization

MCS 563 Lecture 30
Analytic Symbolic Computation
Jan Verschelde, 31 March 2014

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taking derivatives

Suppose $f = f(x, y)$ is reducible: $f = f_1 f_2$.

Applying the product rule for derivatives gives

$$\frac{\partial f}{\partial x} = \frac{\partial f_1}{\partial x} f_2 + f_1 \frac{\partial f_2}{\partial x} = g_1 + g_2$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial f_1}{\partial y} f_2 + f_1 \frac{\partial f_2}{\partial y} = h_1 + h_2.$$

defining $g_1 = \frac{\partial f_1}{\partial x} f_2$, $g_2 = f_1 \frac{\partial f_2}{\partial x}$, $h_1 = \frac{\partial f_2}{\partial y} f_2$, and $h_2 = f_1 \frac{\partial f_2}{\partial y}$.

Then we write the derivatives of $\log(f_1)$ as

$$\frac{\partial}{\partial x}(\log(f_1)) = \frac{1}{f_1}\frac{\partial f_1}{\partial x} = \frac{g_1}{f} \text{ and } \frac{\partial}{\partial y}(\log(f_1)) = \frac{1}{f_1}\frac{\partial f_1}{\partial y} = \frac{h_1}{f}.$$

a partial differential equation

For any p with continuous derivatives, the identity

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}p\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}p\right)$$

holds and we apply it to $p = \log(f_1)$ and $\log(f_2)$ to find

$$\frac{\partial}{\partial x}\left(\frac{h_1}{f}\right) = \frac{\partial}{\partial y}\left(\frac{g_1}{f}\right) \quad \text{and} \quad \frac{\partial}{\partial x}\left(\frac{h_2}{f}\right) = \frac{\partial}{\partial y}\left(\frac{g_2}{f}\right).$$

The partial differential equation

$$\frac{\partial}{\partial y}\left(\frac{g}{f}\right) = \frac{\partial}{\partial x}\left(\frac{h}{f}\right)$$

has nonzero solutions $\Leftrightarrow$ f is reducible.

a criterion for irreducibility

Denote $\deg_x(f)$ (respectively $\deg_y(f)$) as the degree of f when viewed as a polynomial only in x (respectively y).

Theorem (Ruppert's criterion)

A polynomial $f(x,y) \in \mathbb{C}[x,y]$ is irreducible if and only if

$$\frac{\partial}{\partial y}\left(\frac{g}{f}\right) = \frac{\partial}{\partial x}\left(\frac{h}{f}\right)$$

has no nonzero solutions for all polynomial $g, h \in \mathbb{C}[x,y]$, with $\deg_x(g) \leq \deg_x(f) - 1$, $\deg_y(g) \leq \deg_y(f)$, *and* $\deg_x(h) \leq \deg_x(f)$, $\deg_y(h) \leq \deg_y(f) - 2$.

If the condition $\deg_y(h) \leq \deg_y(f) - 2$ on h is changed into $\deg_y(h) \leq \deg_y(f) - 1$, then $g = f_x$ and $h = f_y$ is a solution to the PDE regardless whether f is irreducible or not.

Degree bounds on g_1, g_2, h_1, h_2 give the Ruppert matrix.

an example

Consider for example $f(x,y) = x^2 + y^2 - 1$. Then
$g(x,y) = a_{00} + a_{10}x + a_{01}y + a_{11}xy + a_{02}y^2 + a_{12}xy^2$
and $h(x,y) = b_{00} + b_{10}x + b_{20}x^2$ are
the general forms of the polynomials to satisfy the PDE.

The sequence of commands in Maple generates the Ruppert matrix for f:

```
[> f := x^2 + y^2 - 1;
[> g := sum(sum(a[i,j]*x^i*y^j,
             i=0..degree(f,x)-1),
             j=0..degree(f,y));
[> h := sum(sum(b[i,j]*x^i*y^j,
             i=0..degree(f,x)),
             j=0..degree(f,y)-2);
```

Maple session continued

We setup the Ruppert matrix from the PDE:

```
[> eq := diff(g/f,y) - diff(h/f,x);
[> nq := normal(eq);
[> p := numer(nq);
[> s := coeffs(p,[x,y]);
[> sys := {seq(s[i]=0,i=1..nops([s]))};
[> var := indets(sys);
[> R := LinearAlgebra[GenerateMatrix]
          (sys,var)[1];
[> LinearAlgebra[Rank](R);
```

The rank of the matrix `R` equals 9, which equals the number of columns, so f is indeed irreducible.

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an open problem

Polynomial factorization is important in computer algebra.

One open problem in symbolic-numeric computing is

"*Given is a polynomial* $f(x,y) \in \mathbb{Q}[x,y]$ *and* $\epsilon \in \mathbb{Q}$.
Decide in polynomial time

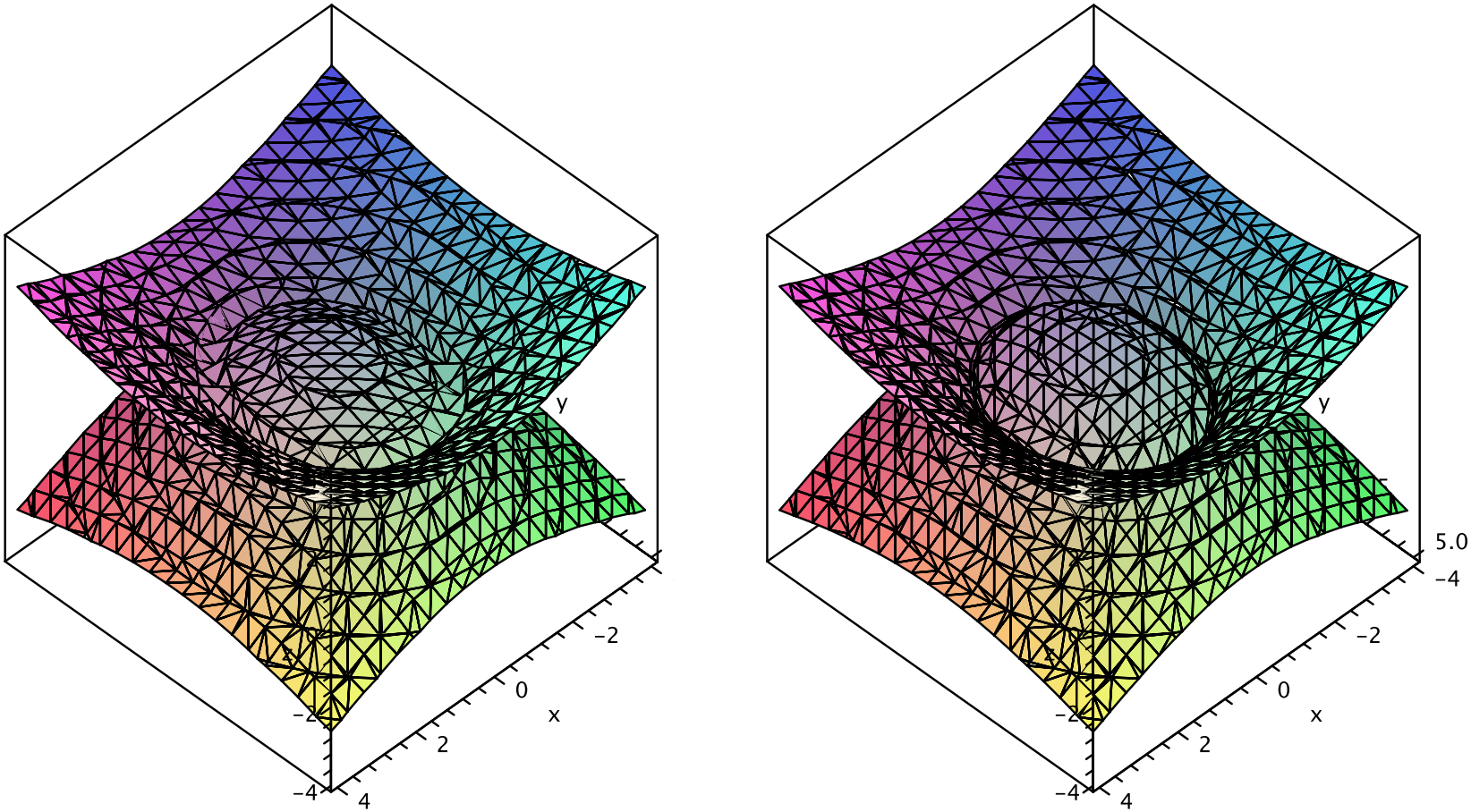
- *in the degree*
- *and coefficient size*

if there is a factorizable $\overline{f}(x,y) \in \mathbb{C}[x,y]$ *with* $||f - \overline{f}|| \leq \epsilon$,
for a reasonable coefficient vector norm $||.||$."

E. Kaltofen: Challenges of symbolic computation: my favorite open problems. *J. Symbolic Computation*, 29(6):891–919, 2000.

an example

In Maple, we apply `implicitplot3d` on `p` directly and execute `factor(p,sqrt(2))` before plotting the factors.



$$p = (9x^2 + 4y^2 + 18\sqrt{2}z^2 - 36)(9x^2 + 4y^2 - 18\sqrt{2}z^2 - 36).$$

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relaxation of the PDE

We assume the polynomial f we have to factor is square free, i.e.: $\mathrm{GCD}(f, f_x) = 1$.

In the PDE, the condition on h is relaxed to $\deg_y(h) \leq \deg_y(f) - 1$ and the PDE is rewritten into

$$f \cdot \left(\frac{\partial g}{\partial y} - \frac{\partial h}{\partial x} \right) + h \cdot \frac{\partial f}{\partial x} - g \cdot \frac{\partial f}{\partial y} = 0.$$

The relaxation of the condition on h implies that the system of linear equations will have at least one solution;
one in the case f is irreducible.
Moreover, the dimension of the solution space equals the number of irreducible factors of f.

Abusing notation, $R(f)$ will still be called the Ruppert matrix, for the matrix resulting of the relaxation on the degree of h.

using the GCD

Given a basis of the null space of the Ruppert matrix,
how do we recover the irreducible factors?

Proposition 1

Let $f = f(x,y)$, $f = f_1 f_2 \cdots f_s$, *and* $g_i = \dfrac{\partial f_i}{\partial x}\dfrac{f}{f_i}$, $i = 1, 2, \ldots, s$.

$$v = \sum_{i=1}^{s} \gamma_i g_i, \quad \gamma_i \neq \gamma_j, i \neq j$$

$$\Rightarrow \quad f_i = \mathrm{GCD}\left(f, v - \gamma_i \frac{\partial f}{\partial x}\right), \quad i = 1, 2, \ldots, s.$$

proof of Proposition 1

Proof. To avoid dot dot dots, we assume $s = 3$.

The g_i's are defined as $g_1 + g_2 + g_3 = \dfrac{\partial f}{\partial x}$. Then:

$$\begin{aligned} v - \gamma_1 \frac{\partial f}{\partial x} &= \gamma_1 g_1 + \gamma_2 g_2 + \gamma g_3 - \gamma_1 (g_1 + g_2 + g_3) \\ &= (\gamma_2 - \gamma_1) g_2 + (\gamma_3 - \gamma_1) g_3 \\ &= (\gamma_2 - \gamma_1) \frac{\partial f_2}{\partial x} f_1 f_3 + (\gamma_3 - \gamma_1) \frac{\partial f_3}{\partial x} f_1 f_2. \end{aligned}$$

Because $\gamma_2 \neq \gamma_1$ and $\gamma_3 \neq \gamma_1$ we find

$$\begin{aligned} \mathrm{GCD}\left(f, v - \gamma_1 \frac{\partial f}{\partial x} \right) &= \mathrm{GCD}\,(f_1 f_2 f_3, \\ &\qquad (\gamma_2 - \gamma_1) \frac{\partial f_2}{\partial x} f_1 f_3 + (\gamma_3 - \gamma_1) \frac{\partial f_3}{\partial x} f_1 f_2 \Big) \\ &= f_1. \end{aligned}$$

The derivations are similar for f_2 and f_3.

form of the basis

The form of the basis elements of the kernel of the Ruppert matrix is described next:

Proposition 2

Consider $f = f(x,y)$ *and* $f = f_1 f_2 \cdots f_s$. *Denote by* $R(f)$ *the Ruppert matrix of the relaxed system linear in the coefficient vectors of the polynomials* g *and* h *with* $\deg(g) \leq (\deg_x(f)-1, \deg_y(f))$ *and* $\deg(h) \leq (\deg_x(f), \deg_y(f)-1)$. *Let* $\mathbf{u} : R(f)\mathbf{u} = \mathbf{0}$, *then* $\mathbf{u} = (\mathbf{v}, \mathbf{w})$, *where* $\mathbf{v}$ *and* $\mathbf{w}$ *are coefficient vectors of the respective polynomials* g *and* h. *Identifying the coefficient vector* $\mathbf{v}$ *with the polynomial* $v(x,y)$ *we have:*

$$v(x,y) = \sum_{i=1}^{s} \gamma_i g_i(x,y), \quad g_i = \frac{\partial f_i}{\partial x}\frac{f}{f_i}, \ i = 1,2,\ldots,s,$$

for some constants $\gamma_i \in \mathbb{C}$.

proof of Proposition 2

Proof. Assuming f is monic, we write f as a function of y, expressing the values for the x-coordinates of f as $x_i(y)$, $i = 1, 2, \ldots, d$, where $d = \deg_x(f)$:

$$f(x(y), y) = \prod_{i=1}^{d} (x - x_i(y)).$$

Since $\deg_x(v) < \deg_x(f)$, we have a partial fraction decomposition

$$\frac{v}{f} = \sum_{i=1}^{d} \frac{a_i(y)}{x - x_i(y)}, \quad a_i(y) = \frac{v(x_i(y), y)}{\prod_{j \neq i} (x - x_i(y))} = \frac{v(x_i(y), y)}{\frac{\partial f}{\partial x}(x_i(y), y)}.$$

We obtain the expression for $a_i(y)$ by equating numerators in the partial fraction decomposition identity for v/f.

partial fraction decompositions

For the polynomial $w(x, y)$ with coefficient vector $\mathbf{w}$,
we also set up a partial fraction decomposition:

$$\frac{w}{f} = \sum_{i=1}^{d} \frac{b_i(y)}{x - x_i(y)} + b_0, \quad b_0 \in \mathbb{C}.$$

Because $\mathbf{u} = (\mathbf{v}, \mathbf{w}) \in \text{kernel}(R(f))$: $\frac{\partial}{\partial y}\left(\frac{v}{f}\right) = \frac{\partial}{\partial x}\left(\frac{w}{f}\right)$.

Applying this property to the partial fraction decompositions:

$$\begin{aligned}
\frac{\partial}{\partial y}\left(\frac{w}{f}\right) &= \sum_{i=1}^{d} \frac{-b_i^2}{(x - x_i(y))^2} \\
\frac{\partial}{\partial x}\left(\frac{v}{f}\right) &= \sum_{i=1}^{d} \frac{1}{x - x_i(y)} \frac{\partial a_i}{\partial y} + \sum_{i=1}^{d} \frac{a_i}{(x - x_i(y))^2} \left(-\frac{\partial x_i}{\partial y}\right).
\end{aligned}$$

So we find that $\frac{\partial}{\partial y}\left(\frac{v}{f}\right) = \frac{\partial}{\partial x}\left(\frac{w}{f}\right)$ implies $\frac{\partial a_i}{\partial y} = 0$.

conclusion of the proof

The constant coefficients a_i belonging to the same factor f_k of f are all conjugated and are all equal, say to γ_k.

So we may write

$$\frac{v}{f} = \sum_{k=1}^{s} \frac{\gamma_k}{\prod_j (x - x_j(y))} = \sum_{k=1}^{s} \gamma_k \frac{\partial f_k}{\partial x} \frac{1}{f_k}.$$

Therefore $v = \sum_{k=1}^{s} \gamma_k \frac{\partial f_k}{\partial x} \frac{f}{f_k} = \sum_{k=1}^{s} \gamma_k g_k$. □

an eigenvalue problem

To recover the g_i's from the linear combinations:

Proposition 3

Let the matrix $V = [\mathbf{v}_1 \mathbf{v}_2 \cdots \mathbf{v}_s]$ *collect the components of the basis vectors of the kernel of the Ruppert matrix* $R(f)$, *i.e.:* $R(f)\mathbf{u} = \mathbf{0}$, $\mathbf{u} = (\mathbf{v}, \mathbf{w})$, $\mathbf{v}$ *contains the coefficient vectors of the polynomials* g *of Ruppert's criterion. For any* $\mathbf{v}$ *in the span of* V, *there is a unique* $A \in \mathbb{C}^{s \times s}$:

$$\mathbf{v}\mathbf{v}_i = \sum_{j=1}^{s} a_{ij} \mathbf{v}_j \frac{\partial f}{\partial x} \mod f.$$

Moreover: $f = \prod_{\substack{\lambda \in \mathbb{C} \\ \det(A - \lambda I) = 0}} \mathrm{GCD}\left(f, v - \lambda \frac{\partial f}{\partial x}\right),$

i.e.: the i*th irreducible factor of* f, $f_i = \mathrm{GCD}\left(f, v - \lambda_i \frac{\partial f}{\partial x}\right)$, *where* λ_i *is the* i*th eigenvalue of* A.

Proof of Proposition 3

Because $\mathbf{v} = \gamma_1 g_1 + \gamma_2 g_2 + \cdots + \gamma_s g_s$, with $g_i = \dfrac{f}{f_i}\dfrac{\partial f}{\partial x}$,
the second statement of the proposition follows immediately
if $\gamma_i = \lambda_i$, as f_i divides $\mathbf{v} - \lambda_i \dfrac{\partial f}{\partial x}$.

Since V is a basis for the null space of the Ruppert matrix, there exists an s-by-s matrix B such that

$$\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_s \end{bmatrix} = B \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_s \end{bmatrix}.$$

computing mod f

We have that $g_i g_j = \left(\frac{\partial f_i}{\partial x} \prod_{\substack{k=1 \\ \neq i}}^{s} f_k \right) \left(\frac{\partial f_j}{\partial x} \prod_{\substack{k=1 \\ \neq j}}^{s} f_k \right)$ is a multiple of f for $i \neq j$, so $g_i g_j \equiv 0 \mod f$. Then we can write

$$\mathbf{v} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_s \end{bmatrix} \equiv B \begin{bmatrix} \mathbf{v} g_1 \\ \mathbf{v} g_2 \\ \vdots \\ \mathbf{v} g_s \end{bmatrix} \equiv B \begin{bmatrix} \lambda_1 g_1^2 \\ \lambda_2 g_2^2 \\ \vdots \\ \lambda_s g_s^s \end{bmatrix} \equiv B \Lambda \begin{bmatrix} g_1^2 \\ g_2^2 \\ \vdots \\ g_s^2 \end{bmatrix} \mod f.$$

$$\text{with} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_s \end{bmatrix}$$

computing mod f continued

The multiplication of $\frac{\partial f}{\partial x}$ with V leads to

$$\frac{\partial f}{\partial x}\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_s \end{bmatrix} \equiv B \begin{bmatrix} \frac{\partial f}{\partial x} g_1 \\ \frac{\partial f}{\partial x} g_2 \\ \vdots \\ \frac{\partial f}{\partial x} g_s \end{bmatrix} \equiv B \begin{bmatrix} g_1^2 \\ g_2^2 \\ \vdots \\ g_s^2 \end{bmatrix} \mod f, \text{ as } \frac{\partial f}{\partial x} = \sum_{i=1}^{s} g_i.$$

So we substitute $$\begin{bmatrix} g_1^2 \\ g_2^2 \\ \vdots \\ g_s^2 \end{bmatrix} \equiv B^{-1} \frac{\partial f}{\partial x} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_s \end{bmatrix} \mod f$$

into the previous derivation for $\mathbf{vv}_i$ and find that $A = B\Lambda B^{-1}$ has eigenvalues λ_i, $i = 1, 2, \ldots, s$.

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outline of the method

1. remove multiple factors with gcd
2. compute $s := \mathrm{Rank}(\mathrm{Null}(R(f))$
   and a basis for $\mathrm{Null}(R(f))$

   return if $s = 1$
3. compute matrix A and its eigenvalues λ_i
4. $f_i := \mathrm{GCD}\left(f, \mathbf{v} - \lambda_i \dfrac{\partial f}{\partial x}\right)$ for $i = 1, 2, \ldots, s$
   and for any $\mathbf{v} \in \mathrm{Null}(R(f))$
5. apply Gauss-Newton on $f - f_1 f_2 \cdots f_s$

Approximate Bivariate Factorization

Input: $f \in \mathbb{C}[x,y]$, $\mathrm{GCD}(f, \frac{\partial f}{\partial x}) = 1$,
f has no approximate factors in $\mathbb{C}[y]$;
$S \subset \mathbb{C}$ and $\#S \geq \deg_x(f) \times \deg_y(f)$.

Output: list of approximate factors of f.

Stage 1: form the Ruppert matrix $R(f)$;
find the last $\deg(f)+1$ singular values σ_i of $R(f)$,
$\sigma_n \geq \sigma_{n-1} \geq \cdots \geq \sigma_2 \geq \sigma_1$;
let s be the index so σ_{s+1}/σ_s is maximal;
if $s = 1$, then return f;
form a basis $\mathbf{v}_1, \mathbf{v}_1, \ldots, \mathbf{v}_s$ from the last
s right singular vectors of $R(f)$;

stages 2 and 3

Stage 2: $\mathbf{v} := \sum_{s_i \in S} s_i \mathbf{v}_i$, with coefficients s_i
selected uniformly and independently;

for $y = \alpha$, compute a_{ij} that minimize

$$\left\| \text{remainder}\left(\mathbf{v}\mathbf{v}_i - \sum_{j=1}^{s} a_{ij}\mathbf{v}_j \frac{\partial f}{\partial x}, f \right) \right\|_2 ;$$

compute the eigenvalues λ_i of $A = [a_{ij}]$;

Stage 3: $f_i := \text{GCD}\left(f, \mathbf{v} - \lambda_i \frac{\partial f}{\partial x} \right)$, for $i = 1, 2, \ldots, s$,
where GCD is an approximate GCD.

Summary + Exercises

Numerical rank via SVD, least squares, eigenvalues, and approximate GCD are key to a symbolic-numeric algorithm for approximate bivariate factorization.

Exercises:

1. Find a general formula for the size of the Ruppert matrix, in terms of the degrees $\deg_x(f)$ and $\deg_y(f)$.
2. Show that for $f = f(x,y)$, $f = f_1 f_2$, $g_1 = \frac{\partial f_1}{\partial x} f_2$, $g_2 = f_1 \frac{\partial f_2}{\partial x}$, $h_1 = \frac{\partial f_2}{\partial y} f_2$, and $h_2 = f_1 \frac{\partial f_2}{\partial y}$:

$$f\left(\frac{\partial g_1}{\partial y} - \frac{\partial h_1}{\partial x}\right) + h_1 \frac{\partial f}{\partial x} - g_1 \frac{\partial f}{\partial y} = 0.$$

more exercises

3. Download the Maple code at

`http://www4.ncsu.edu/~kaltofen/software/appfac/issac04_mws/multifac_1.3.mpl` and use it to factor

$$f(x,y) = 9 + 23y^2 + 13yx^2 + 6y + 7y^3 + 13y^2x^2 + x^4 + 6yx^4 + x^6.$$

4. Download `ApaTools` available via the homepage of Zhonggang Zeng and use it to factor f from the previous exercise.
5. Consider f from above, but now add some random errors to the coefficients, of magnitude 10^{-k}, for k ranging from 1 to 14. For $k = 1$, f is irreducible, while for $k = 14$, the numerical algorithm should return the same factorization as in the exact case. For which k is f no longer irreducible?