Binomial Systems

1. Binomial Ideals
   - definition and properties
   - solving a zero dimensional pure difference ideal
2. Commuting Birth-and-Death Processes
   - models from ecology and queuing theory
   - a system of quadratic polynomials
3. Cellular decompositions
   - decompositing binomial ideals
4. Using Macaulay2
   - running examples with the package Binomials

MCS 563 Lecture 36
Analytic Symbolic Computation
Jan Verschelde, 14 April 2014

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binomial ideals

Consider $\mathbb{K}[\mathbf{x}]$, with $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$.

Typically we will assume that $\mathbb{K}$ is algebraically closed, so $\mathbb{K} = \mathbb{C}$ is our default coefficient field. Then

$$I = \langle\, c_{\mathbf{a}} x^{\mathbf{a}} - c_{\mathbf{b}} x^{\mathbf{b}} \mid \mathbf{a}, \mathbf{b} \in \mathbb{N}^n, c_{\mathbf{a}}, c_{\mathbf{b}} \in \mathbb{K}^* \,\rangle$$

is a binomial ideal. A polynomial is a binomial if it has exactly two monomials with a nonzero coefficient.

A binomial ideal is generated by binomials.

Definition

A *pure difference ideal* is an ideal generated by differences of monic monomials, i.e.: all generators are of the form $\mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{b}}$.

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solving a zero dimensional pure difference ideal

Proposition

Let I be a zero dimensional pure difference ideal. There is a primitive root of unity ξ, such that all complex solutions of I are contained in the cyclotomic field $\mathbb{Q}(\xi)$.

Proof. Let $\mathcal{G}$ be a lexicographic Gröbner basis.

- Because all *S*-polynomials are pure difference binomials, $\mathcal{G}$ consists of pure difference binomials.
- As the ideal is zero dimensional and because a lexicographic order eliminates, at least one of the binomials in $\mathcal{G}$ is univariate.
- The solutions of the univariable equations exists in a cyclotomic field. By substituting the solution for that variable in the other equations, an univariate equation in another variable is obtained.
- After extending the partial solutions, all roots of unity encountered during univariate solving define $\mathbb{Q}(\xi)$ where the solutions live. $\square$

toric varieties

Because the exponents determine the structure of the ideal, we then define a toric ideal as

$$I_A = \left\langle\, \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \mid \mathbf{u}, \mathbf{v} \in \mathbb{N}^n \text{ and } A\mathbf{u} = A\mathbf{v} \,\right\rangle.$$

The solution set of a toric ideal is a toric variety.

As an alternative to the ideal description,
a toric variety over $\mathbb{C}$ is defined as

- a complex algebraic variety with an action of $(\mathbb{C}^*)^n$ and
- a dense open subset isomorphic to $(\mathbb{C}^*)^n$ carrying the regular action.

That is: a toric variety is an algebraic torus closure.

In polyhedral homotopies: at ∞ and at 0 are equivalent.

binomial primary decompositions

Binomial ideals have special properties, for instance:

Theorem (Theorem 2.6 in [Eisenbud-Sturmfels, 1996])

If $\mathbb{K}$ is algebraically closed and I is a binomial ideal in $\mathbb{K}[\mathbf{x}]$, then every associated prime of I is generated by binomials.

The condition that $\mathbb{K}$ is algebraically closed is essential:

over $\mathbb{Q}$: $\langle x^3 - 1 \rangle = \langle x - 1 \rangle \cap \langle x^2 + x + 1 \rangle$.

If we extend $\mathbb{Q}$ with $w = e^{(2\pi\sqrt{-1})/3}$, then over $\mathbb{Q}(w)$:

$\langle x^3 - 1 \rangle = \langle x - 1 \rangle \cap \langle x + (1 - \sqrt{-3})/2 \rangle \cap \langle x + (1 + \sqrt{-3})/2 \rangle$.

Binomial Systems

Commuting Birth-and-Death Processes

An application of combinatorics and algebraic statistics:

- Models arising in ecology and queuing theory study population sizes and numbers of individual waiting in a queue.
- Markov chains are described by tridiagonal transition matrices P, $P(i,j)$ is the probability of going from step i to j.
- In a higher-dimensional model the state space is a product of intervals in higher-dimensional lattices, e.g.:
  - ecology: keep track of the type of individuals in a population;
  - queuing: several servers have each their own set of customers.
- The mathematical tools are
  - one dimension: orthogonal polynomials;
  - higher dimension: binomial primary decomposition.

an example in dimension two

Define a grid $E = \{0, 1, \ldots, m\} \times \{0, 1, \ldots, n\}$
where (i, j) is connected to $(k, \ell) \Leftrightarrow |i - k| + |j - \ell| = 1$.

The transition probabilities are

$$\begin{aligned}
\text{go left: } L_{i,j} &= \mathrm{prob}\{Z_{k+1} = (i-1, j) \mid Z_k = (i, j)\} \\
\text{go right: } R_{i,j} &= \mathrm{prob}\{Z_{k+1} = (i+1, j) \mid Z_k = (i, j)\} \\
\text{go down: } D_{i,j} &= \mathrm{prob}\{Z_{k+1} = (i, j-1) \mid Z_k = (i, j)\} \\
\text{go up: } U_{i,j} &= \mathrm{prob}\{Z_{k+1} = (i, j+1) \mid Z_k = (i, j)\}
\end{aligned}$$

Commuting relations:

$$\begin{aligned}
U_{i,j} R_{i,j+1} &= R_{i,j} U_{i+1,j} && \text{(up-right)} \\
D_{i,j+1} R_{i,j} &= R_{i,j+1} D_{i+1,j+1} && \text{(down-right)} \\
D_{i+1,j+1} L_{i+1,j} &= L_{i+1,j+1} D_{i,j+1} && \text{(down-left)} \\
U_{i+1,j} L_{i+1,j+1} &= L_{i+1,j} U_{i,j} && \text{(up-left)}
\end{aligned}$$

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a system of quadratic polynomials

$$E = \{0,1,\ldots,n_1\} \times \{0,1,\ldots,n_2\} \times \ldots \times \{0,1,\ldots,n_m\}$$

For all pairs (i,j): $1 \leq i < j \leq m$, the commuting requirement

$$\begin{array}{l}
P(u,u+e_i)P(u+e_i,u+e_i+e_j) - P(u,u+e_j)P(u+e_j,u+e_i+e_j), \\
P(u,u+e_i)P(u+e_i,u+e_i-e_j) - P(u,u-e_j)P(u+e_j,u+e_i-e_j), \\
P(u,u-e_i)P(u-e_i,u-e_i+e_j) - P(u,u+e_j)P(u+e_j,u-e_i+e_j), \\
P(u,u-e_i)P(u-e_i,u-e_i-e_j) - P(u,u-e_j)P(u+e_j,u-e_i-e_j)
\end{array}$$

is a system of quadratic polynomials in the unknowns $P(u,v)$.

the ideal of commuting birth-and-death processes

Denote by $I^{(n_1,n_2,\ldots,n_m)}$ the ideal generated by the quadratic polynomials in the commuting requirement.

In the two dimensional case, $I^{(m,n)}$ is generated by $4mn$ quadratic binomials, for (i,j): $0 \leq i < m$ and $0 \leq j < n$:

$$\begin{aligned}
U_{i,j}R_{i,j+1} - R_{i,j}U_{i+1,j} &= 0, \\
R_{i,j+1}D_{i+1,j+1} - D_{i,j+1}R_{i,j} &= 0, \\
D_{i+1,j+1}L_{i+1,j} - L_{i+1,j+1}D_{i,j+1} &= 0, \\
L_{i+1,j}U_{i,j} - U_{i+1,j}L_{i+1,j+1} &= 0.
\end{aligned}$$

the smallest example

The possibilities that

$$\begin{pmatrix} 0 & 0 & R_{0,0} & 0 \\ 0 & 0 & 0 & R_{0,1} \\ L_{1,0} & 0 & 0 & 0 \\ 0 & L_{1,1} & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & U_{0,0} & 0 & 0 \\ D_{0,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & U_{1,0} \\ 0 & 0 & D_{1,1} & 0 \end{pmatrix}$$

commute are revealed by the primary decomposition of

$$I^{(1,1)} = \left\langle \begin{array}{c} U_{0,0}R_{0,1} - R_{0,0}U_{1,0} \\ R_{0,1}D_{1,1} - D_{0,1}R_{0,0} \\ D_{1,1}L_{1,0} - L_{1,1}D_{0,1} \\ L_{1,0}U_{0,0} - U_{1,0}L_{1,1} \end{array} \right\rangle .$$

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cellular ideals

Consider $\mathcal{E} \subseteq \{1, 2, \ldots, n\}$ and
denote the algebraic torus corresponding to $\mathcal{E}$ by

$$(\mathbb{K}^*)^{\mathcal{E}} = \{ \mathbf{x} \in \mathbb{K}^n \mid x_i \neq 0 \text{ for } i \in \mathcal{E} \text{ and } x_j = 0 \text{ for } j \notin \mathcal{E} \}.$$

The central definition is

Definition

A proper binomial ideal I in $\mathbb{K}[\mathbf{x}]$ is *cellular* if each variable x_i is either a nonzerodivisor or nilpotent modulo I.

Primary ideals I are cellular as every element in $\mathbb{K}[\mathbf{x}]/I$ is either nilpotent or a nonzerodivisor.

characterizing cellular ideals

We have a characterization for an ideal I being cellular in the following lemma.

Lemma

A proper binomial ideal I in $\mathbb{K}[\mathbf{x}]$ is cellular if and only if there exists a set $\mathcal{E} \subseteq \{1, 2, \ldots, n\}$ of indices of variables in $\mathbf{x}$ such that

1. $I = \left(I : \left(\prod_{i \in \mathcal{E}} x_i \right)^\infty \right)$; *and*
2. *For every $i \not\in \mathcal{E}$, there exists an integer $d_i \geq 0$ such that $\langle x_i^{d_i} \mid i \not\in \mathcal{E} \rangle$ is contained in I.*

an algorithm

The `Binomials` package in Macaulay 2 provides an implementation of the following recursive algorithm:

Algorithm [cellular decomposition]

Input: a binomial ideal I.
Output: a cellular decomposition of I.

1. If I is cellular, then return I.
2. Choose x_i that is a zerodivisor but not nilpotent modulo I.
3. Determine the power m such that $(I : x_i^m) = (I : x_i^\infty)$.
4. Call the algorithm on $(I : x_i^m)$ and $I + \langle x_i^m \rangle$.

solving toric ideals

Input: a zero dimensional toric ideal I.
Output: roots of unity to extend $\mathbb{Q}$ and solutions in $V(I)$.

1. Compute a cellular decomposition of I.
2. For each cellular component do

2.1 Set the noncell variables to zero and determine the product $D := \prod_{i \notin \mathcal{E}} d_i$ of the minimal powers of the noncell variables.

2.2 Compute a lexicographic Gröbner basis and solve the lattice ideal of the cellular component, adjoining roots of unity.

2.3. Save each solution D times.

3. Compute the least common multiple m of the powers of the adjoined roots of unity and construct the cyclotomic field $\mathbb{Q}(w_m)$.
4. Return the list of solutions as elements in $\mathbb{Q}(w_m)$.

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running Binomials

The package `Binomials` of Thomas Kahle is in Macaulay2.

```
i1 : S = QQ[x,y,z];
i2 : I = ideal(x^2-y,y^3-z,x*y-z);
i3 : loadPackage "Binomials";
i4 : binomialSolve I
BinomialSolve created a cyclotomic field
of order 3

o4 = {{1, 1, 1}, {- ww  - 1, ww , 1},
                       3       3
    {ww , - ww  - 1, 1},
       3      3
    {0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
i5 : degree I
o5 = 6
```

binomial primary decomposition

```
i6 : BPD I
Running cellular decomposition:
cellular components found: 1
cellular components found: 2
Decomposing cellular components:
Decomposing cellular component: 1 of 2
1 monomial to consider for this cellular component
BinomialSolve created a cyclotomic field of order 3
done
Decomposing cellular component: 2 of 2
3 monomials to consider for this cellular component
done
Removing redundant components...
4 Ideals to check
3 Ideals to check
2 Ideals to check
1 Ideals to check
0 redundant ideals removed.
Computing mingens of result.
```

the result

The primary decomposition of $\langle x^2 - y, y^3 - z, xy - z \rangle$ is

```
o6 = {ideal (z - 1, y - 1, x - 1),

   ideal (z - 1, y - ww , x + ww  + 1),
                       3         3

   ideal(z - 1, y + ww  + 1, x - ww ),
                      3             3


                2        2
   ideal (z, y , x*y, x  - y)}
```

associated prime

We consider the last ideal in the primary decomposition

```
i7 : I = ideal(z,y^2,x*y,x^2 - y);

i8 : binomialAssociatedPrimes I
3 monomials to consider for this cellular
component

o8 = {ideal (z, y, x)}
```

cellular decompositions

```
i2 : S = QQ[x1,x2,x3,x4,x5];
i3 : I = ideal(x1*x4^2-x2*x5^2,
 x1^3*x3^3-x2^4*x4^2, x2*x4^8-x3^3*x5^6);
i4 : I
                 2         2    3  3     4  2
o4 = ideal (x1*x4  - x2*x5 , x1 x3  - x2 x4 ,
          8     3  6
     x2*x4  - x3 x5 )

i5 : BCD I
```

the output of BCD I

```
cellular components found: 1
redundant component
redundant component
cellular components found: 2
                  2        2    3  3     4  2
o5 = {ideal (x1*x4  - x2*x5 , x1 x3  - x2 x4 ,
    3  4     2  3  2
  x2 x4  - x1 x3 x5 ,
   2  6      3  4      2  3  2
 x2 x4  -  x2 x4  - x1 x3 x5 ,
   2  6         3  4       8     3  6
 x2 x4  -  x1*x3 x5 , x2*x4  - x3 x5 ),
           2       2        2    5
 ideal (x1 , x1*x4  - x2*x5 , x2 ,
    6    4  2    8
  x5 , x2 x4 , x4 )}
```

Summary + Exercises

Binomial ideals are an interesting class of problems.

Exercises:

1. Explore the package Binomials in Macaulay2.
2. Explore the capabilities in CoCoA for handling binomial ideals.
3. Explore the capabilities in Sage/Singular for binomial ideals.