Border Bases

1. Zero Dimensional Ideals
   - computing in the quotient ring

2. Exploring Oil Fields
   - solving an interpolation problem
   - an ill-conditioned Gröbner basis

3. Border Bases
   - definition and computation

MCS 563 Lecture 25
Analytic Symbolic Computation
Jan Verschelde, 12 March 2014

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zero dimensional ideals

Given a polynomial system $f(\mathbf{x}) = 0$, the ideal $I = \langle f \rangle$ is zero dimensional if its solution set $V(I) = f^{-1}(\mathbf{0})$ consists of finitely many isolated points.

For zero dimensional ideals, the dimension of the quotient ring $R[I] = \mathbb{C}[\mathbf{x}]/I$ equals the cardinality of the solution set.

The quotient ring $R[I]$ of a ideal I is

$$R[I] = \{ \ [p]_I \mid p \in \mathbb{C}[\mathbf{x}] \ \} \text{ is the residue class ring,}$$

with $[p]_I$ the residue class of p mod I:

$$\begin{aligned} [p]_I &= \{ \ r \in \mathbb{C}[\mathbf{x}] \mid p - r \in I \ \} \\ &= \{ \ r \in \mathbb{C}[\mathbf{x}] \mid p(\mathbf{z}) = r(\mathbf{z}), \forall \mathbf{z} \in V(I) \ \}. \end{aligned}$$

an example

Consider the system

$$f(x,y) = \begin{cases} x^2 + 4xy + 4y^2 - 4 = 0 \\ 4x^2 - 4xy + y^2 - 4 = 0 \end{cases} \qquad D = 2 \times 2.$$

According to Bézout's theorem, we expect four solutions.

Row reduction on the coefficients leads to

$$\begin{cases} 15x^2 - 20xy - 12 = 0 \\ 20y^2 + 20xy - 12 = 0. \end{cases}$$

A natural basis for the quotient ring $\mathbb{C}[x,y]/\langle f \rangle$
is $\mathbf{b} = (1, x, y, xy)$.

multiplication matrices

For $\mathbf{b} = (1, x, y, xy)$ we use $15x^2 - 20xy - 12 = 0$
to rewrite x^2 and x^2y modulo the ideal:

$$
\begin{aligned}
x\mathbf{b} &= x \begin{pmatrix} 1 \\ x \\ y \\ xy \end{pmatrix} = \begin{pmatrix} x \\ x^2 \\ xy \\ x^2y \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{12}{15} & 0 & 0 & \frac{20}{15} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{48}{125} & \frac{36}{125} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \\ xy \end{pmatrix}
\end{aligned}
$$

Observe: $A_x\mathbf{b} = x\mathbf{b}$ is an eigenvalue problem.

commuting families

The multiplication matrices A_x and A_y
have joint eigenvectors:

$$A_x \mathbf{b} = x \mathbf{b} \quad \text{and} \quad A_y \mathbf{b} = y \mathbf{b}.$$

This implies

$$\begin{aligned} A_y A_x \mathbf{b} &= x A_y \mathbf{b} = xy \mathbf{b} \text{ and} \\ A_x A_y \mathbf{b} &= y A_x \mathbf{b} = yx \mathbf{b}, \end{aligned}$$

so $A_x A_y = A_y A_x$.

The multiplication matrices form a commuting family.
The relations $A_x A_y - A_y A_x = 0$ are helpful for numerical methods on approximate input data.

duality

The dual space of the quotient ring $R[I]$:

$$\begin{aligned}(R[I])^* \;=\; \{ \, l : R[I] \rightarrow \mathbb{C} : [p]_I \mapsto l(p) := l(r), \\ r \in R[I], p - r \in I \, \}.\end{aligned}$$

The dual of I is

$$D[I] = \{ \, \ell : \mathbb{C}[\mathbf{x}] \rightarrow \mathbb{C} : l(p) = 0, \forall p \in I \, \}.$$

Theorem

$D[I] = (R[I])^*$ and $R[I] = (D[I])^*$.

For a vector space V, if $\mathbf{b}$ is a basis for V and $\mathbf{c}$ a basis for its dual V^*, then $\mathbf{c}^T\mathbf{b}$ equals the identity matrix.

the vanishing ideal

The ideal of the dual:

$$I[D[I]] = \ker(D[I]) = \{ \, p \in \mathbb{C}[\mathbf{x}] : \ell(p) = 0, \forall \ell \in D[I] \, \} = I.$$

Note that this relation is more satisfactory than $I[Z[I]] = \sqrt{I}$.

The dual defines the multiplicity structure of a multiple zero.

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exploring oil fields

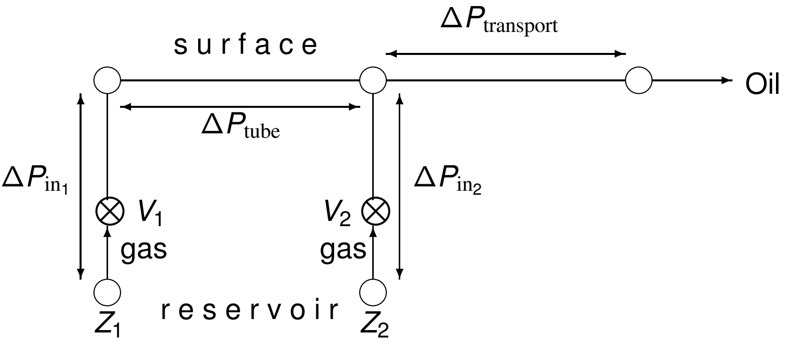
A reservoir consists of three layers: water, oil, and gas,
with water at the bottom and gas on top
(because of their densities).

Without a rock to cap the escape of gas, there can be
no reservoir, so drilling is needed to extract the oil.

Gas flows to the surface by itself and often the oil has enough pressure
to come up as well.

For the exploration, pressure levels are critical
to recover as much oil as possible.

an oil field with two wells



Schematic representation of a well with two zones Z_1, Z_2, two pipes with valves V_1, V_2.

The empty circles are pressure readers.

We observe four pressure differences:

$$\Delta P_{\text{in}_1}, \Delta P_{\text{in}_2}, \Delta P_{\text{tube}}, \text{ and } \Delta P_{\text{transport}}.$$

an algebraic model

To link the gas production with pressure differences, we associate five variables to the physical quantities:
$x_1 = \Delta P_{\mathrm{in}_1}$, $x_2 = \Delta P_{\mathrm{in}_2}$, x_3 is the amount of gas produced, $x_4 = \Delta P_{\mathrm{tube}}$, and $x_5 = \Delta P_{\mathrm{transport}}$.

These five quantities are assumed as the driving forces in the production of oil.

The problem is to determine a polynomial $f \in \mathbb{R}[x_1, x_2, \ldots, x_5]$ based on measured values for the gas production, pressure differences, and the oil production.

The result of the algebraic model is a polynomial f which explains the oil production in function of the driving forces.

A direct approach for this problem is to apply a least squares approximation on the given data, but this approach ignores the relations between the variables.

rewriting polynomials

Given is a set of approximate data points $\mathbb{X} \subset \mathbb{R}^5$, normalized properly so $\mathbb{X} \subset [-1, +1]^5$. For exact points (or mapping $\mathbb{X}$ into $\mathbb{Q}$), the Buchberger-Möller algorithm produces a Gröbner basis G for the vanishing ideal.

Using the division algorithm to write any polynomial p as

$$p = \sum_{g \in G} q_g g + f, \quad q_g \in \mathbb{Q}[\mathbf{x}]$$

means that we take into account the relations between the variables, relations imposed by the given data points.

As $g \in G$ vanishes at any point, the interpolating polynomial has its shape determined by the monomials in the remainder f. Our f in the algebraic model has the form

$$f = \sum_{\mathbf{x}^\mathbf{a} \not\in \mathrm{in}_>(G)} c_\mathbf{a} \mathbf{x}^\mathbf{a}, \quad c_\mathbf{a} \in \mathbb{R}.$$

The problem is then to find a suitable monomial basis for f.

ordering variables

While the problem with Gröbner bases for approximate data is their sensitivity to small perturbations, we end by pointing out that the order of the variables still matters.

Calculations reported in [Kreuzer, Poulisse, Robbiano 2009] with degree reverse lexicographical term order result in a polynomial with a dominant presence of x_5.

This dominance of x_5 in the model is unfortunate as x_5 is the pressure difference in the transport pipe on the surface and most unlikely to be a key factor.

Therefore, a relabeling of the variables according to the physical hierarchy leads to an interpolating polynomial more consistent with the physical interpretation.

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the example of Windsteiger

$$f(x,y) = \begin{cases} -4 + 3\left(\frac{172966043}{174178537}x - \frac{42176556}{358072327}y\right)^2 \\ \quad + \left(1/3 + \frac{42176556}{358072327}x + \frac{172966043}{174178537}y\right)^2 = 0 \\ -4 + \left(1/3 - \frac{42176556}{358072327}y + \frac{172966043}{174178537}x\right)^2 \\ \quad + 4\left(\frac{172966043}{174178537}y + \frac{42176556}{358072327}x\right)^2 = 0. \end{cases}$$

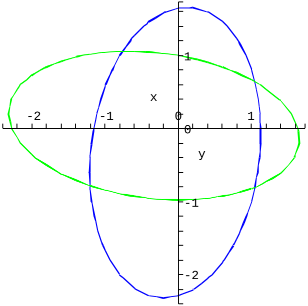
After `expand` and `evalf` on the system: $\widetilde{f}(x,y) =$

$$\begin{cases} -3.888888889 + 2.972252063x^2 - 0.4678714642xy \\ +1.027747937y^2 + 0.07852520812x + 0.6620258576y = 0 \\ -3.888888889 - 0.07852520812y + 0.6620258576x \\ +3.958378094y^2 + 0.7018071964xy + 1.041621906x^2 = 0. \end{cases}$$

The coefficients of the system are well scaled.

looking at two curves

Small changes in the curves leads to small shifts in the intersection points:



Eliminate y and the x-values of the solutions are clustered.
Instead of taking the basis $(1, x, x^2, x^3)$,
the basis $(1, x, x^2, y)$ gives better results.

Border Bases

border bases

Given an ideal I, a normal set $N[I]$ is a set of monomials $N[I] = \{ \mathbf{x}^{\mathbf{a}} \mid \mathbf{a} \in A \}$ such that

1. for all $\mathbf{x}^{\mathbf{a}} \in N[I]$: if $\mathbf{x}^{\mathbf{a}'}$ divides $\mathbf{x}^{\mathbf{a}}$, then $\mathbf{x}^{\mathbf{a}'} \in N[I]$; and
2. $B = \{ [\mathbf{x}^{\mathbf{a}}]_I \mid \mathbf{x}^{\mathbf{a}} \in N[I] \}$ is a basis for $R[I]$.

The normal set and the multiplication matrices provide a normal form for every polynomial modulo the ideal.

Given a normal set N, the border set is $B[N] = \{ \mathbf{x}^{\mathbf{a}} \mid \mathbf{x}^{\mathbf{a}}/x_k \in N, \text{ for some } k \}$.

The border set collects all leading monomials in a border basis, defined by $A_{x_k}\mathbf{b} = x_k\mathbf{b}$, for all $k = 1, 2, \ldots, n$, and where $\mathbf{b}$ is $N_<$, the ordered normal set.

The existence of a border basis is implied by a Gröbner basis, i.e.: given a Gröbner basis for a zero dimensional ideal, we can directly write down a border basis.

computational methods

To compute a border basis, several methods are proposed:

- the mixed volume = the size of the border basis

  Fill in the normal set starting at the lowest degree monomials. Use the original equations to rewrite monomials applying pivoting.

- algebraic algorithms

  Use a Buchberger criterion for a border prebasis.

- stable normal forms

  Link normal form algorithm to commuting relations.

approximate ideals and border bases

Given a set of points $\mathbb{X} \subset \mathbb{R}^n$,
an ϵ-approximate vanishing ideal I of $\mathbb{X}$
is a system of generators f_i, $i = 1, 2, \ldots, N$ of I such that

- $||f_i|| = 1$ for all i and
- $|f_i(\mathbf{z})| < \epsilon$ for all $\mathbf{z} \in \mathbb{X}$.

A set G is an ϵ-approximate border basis for the ideal I generated by G if

$$\text{for all } g_i, g_j \in G : ||S(g_i, g_j)|| < \epsilon,$$

where $S(g_i, g_j)$ is the S-polynomial of g_i and g_j.

application of SVD

With the singular value decomposition (SVD) we compute the kernel of the monomials in the normal set, evaluated at the points in $\mathbb{X} \subset [-1, +1]^n$.

The normal set is constructed incrementally.

Applying row reduction with pivoting the algorithm constructs polynomials that vanish ϵ-approximately on $\mathbb{X}$.

The algorithm is the numerical analogue to the Buchberger-Möller algorithm.

Summary + Exercises

Border bases lead to numerically stable normal forms.

Exercises:

1. Compute a lexicographic Gröbner basis for the example of Windsteiger and use it to compute the roots. Compare the results of the eliminating orders $x > y$ and $y > x$.
2. Apply the Maple commands `Groebner[SetBasis]` and `Groebner[MulMatrix]` to a Gröbner basis for the example of Windsteiger.
3. Download and install SYNAPS [Mourrain et al. 2008] on your computer and compute some examples of border bases. Compare the running time with the computation of a Gröbner basis in any computer algebra system.