Kushnirenko’s Theorem

1. Binomial Systems
   - unimodular coordinate transformations
2. Design of 4-bar Mechanisms
   - Chebyshev’s straight line mechanism
3. Regular Triangulations
   - support sets span Newton polytopes
   - the theorem of Kushnirenko
4. Polyhedral Algorithms
   - the Hermite normal form
   - placing points into a regular triangulation

MCS 563 Lecture 16
Analytic Symbolic Computation
Jan Verschelde, 19 February 2014

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Binomial Systems

The sparsest nontrivial polynomial systems have exactly two monomials with nonzero coefficient in every equation.
Such systems are called binomial systems.

Denote $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.
A matrix A with integer coefficients and a vector $\mathbf{c} \in (\mathbb{C}^*)^n$ define a *binomial system*, denoted as $\mathbf{x}^A = \mathbf{c}$.
The columns of A define the exponent vectors of the equations in the binomial system, i.e.: for $A = [\mathbf{a}_1 \mathbf{a}_2 \cdots \mathbf{a}_n]$, we have

$$\mathbf{x}^A = \mathbf{x}^{[\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n]} = [\mathbf{x}^{\mathbf{a}_1} \ \mathbf{x}^{\mathbf{a}_2} \ \ldots \ \mathbf{x}^{\mathbf{a}_n}]$$

and

$$\mathbf{x}^A = \mathbf{c} \Leftrightarrow [\mathbf{x}^{\mathbf{a}_1} = c_1 \ \mathbf{x}^{\mathbf{a}_2} = c_2 \ \cdots \ \mathbf{x}^{\mathbf{a}_n} = c_n].$$

an example

Consider for example the following binomial system:

$$\left\{ \begin{array}{l} x_1^3 x_2^2 = 1 \\ x_1^5 x_2^1 = 1 \end{array} \right. \quad \Leftrightarrow \quad \mathbf{x}^A = \mathbf{c} : [x_1 \;\; x_2]^{\left[\begin{array}{cc} 3 & 5 \\ 2 & 1 \end{array} \right]} = [1 \;\; 1].$$

To make A upper triangular, we take its first column and compute $\gcd(3,2) = 1 = (+1) \cdot 3 + (-1) \cdot 2$.
We define a unimodular matrix M:

$$M = \left[\begin{array}{rr} 1 & -1 \\ -2 & 3 \end{array} \right], \quad \det(M) = 1$$

$$MA = \left[\begin{array}{rr} 1 & -1 \\ -2 & 3 \end{array} \right] \left[\begin{array}{cc} 3 & 5 \\ 2 & 1 \end{array} \right] = \left[\begin{array}{rr} 1 & 4 \\ 0 & -7 \end{array} \right] = U.$$

coordinate transformations

The matrix M defines a change of coordinates:

$$\mathbf{y}^M = \mathbf{x} \Rightarrow \mathbf{x}^A = \mathbf{y}^{MA} = \mathbf{y}^U.$$

More explicitly, the coordinate transformation is

$$[x_1 \;\; x_2] = [y_1 \;\; y_2]^{\left[\begin{array}{rr} 1 & -1 \\ -2 & 3 \end{array}\right]} = [x_1 = y_1 y_2^{-2} \;\; x_2 = y_1^{-1} y_2^3].$$

Transformation of coordinates $=$ matrix multiplication MA:

$$\left\{ \begin{array}{l} x_1^3 x_2^2 = \left(y_1 y_2^{-2}\right)^3 \left(y_1^{-1} y_2^3\right)^2 = y_1^{1\cdot 3 + (-1)\cdot 2} y_2^{(-2)\cdot 3 + 3\cdot 2} = y_1 \\ x_1^5 x_2^1 = \left(y_1 y_2^{-2}\right)^5 \left(y_1^{-1} y_2^3\right)^1 = y_1^{1\cdot 5 + (-1)\cdot 1} y_2^{(-2)\cdot 5 + 3\cdot 1} = y_1^4 y_2^{-7}. \end{array} \right.$$

solving binomial systems

Because $\det(M) = 1$, the number of solutions is unchanged:

$$|\det(U)| = |\det(MA)| = |\det(U)\det(A)| = |\det(A)| = 7.$$

The upper triangular U is the Hermite normal form of A.

Solving $\mathbf{x}^A = \mathbf{c}$ is reduced to solving $\mathbf{y}^U = \mathbf{c}$, $\mathbf{x} = \mathbf{y}^M$.

Theorem (regularity of a binomial system)

If $\det(A) \neq 0$ *and* $\mathbf{c} \in (\mathbb{C}^*)^n$, *then* $\mathbf{x}^A = \mathbf{c}$ *has exactly as many as* $|\det(A)|$ *isolated regular solutions in* $(\mathbb{C}^*)^n$.

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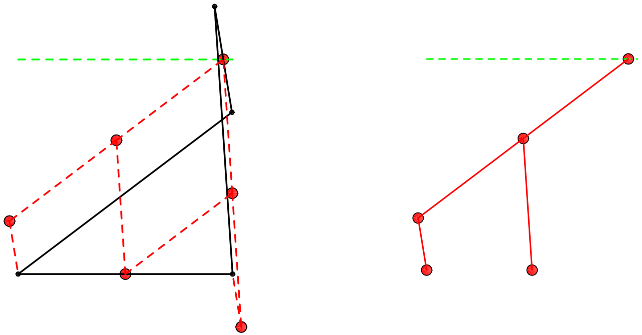

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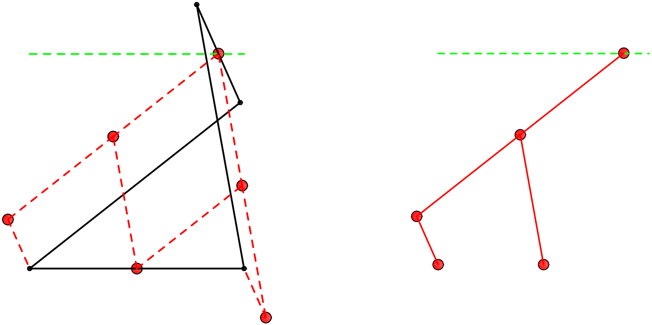

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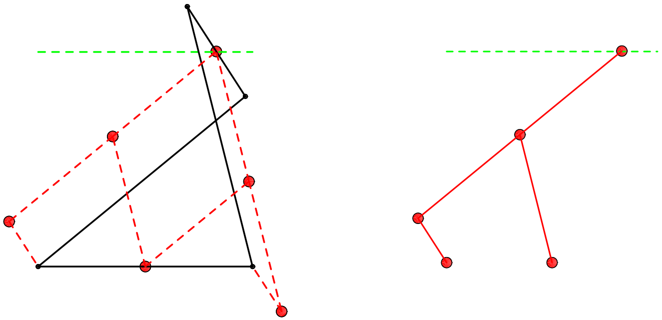

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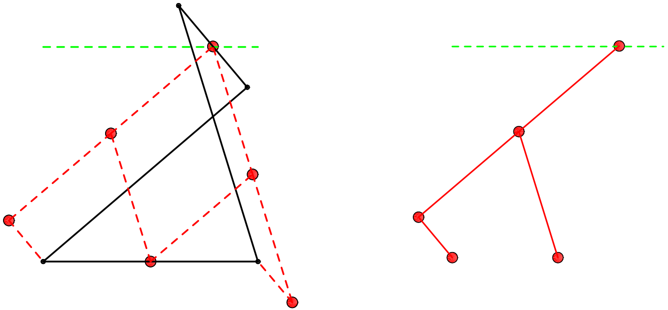

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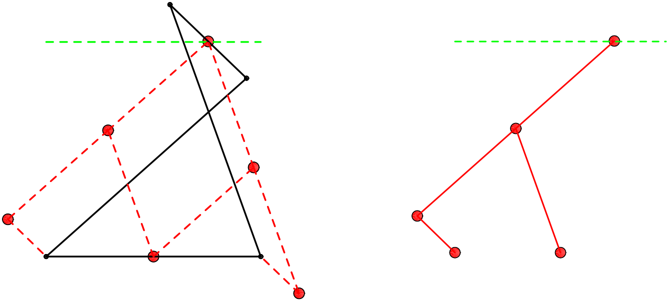

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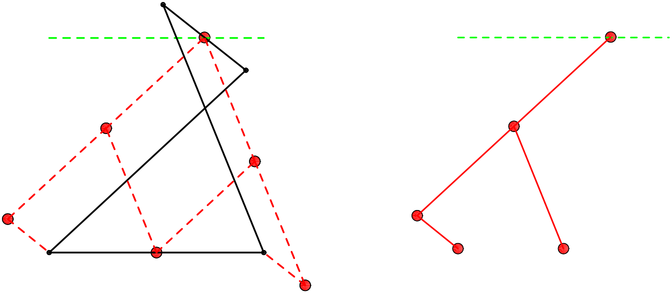

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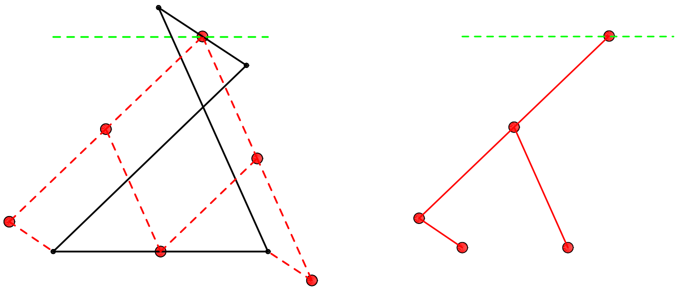

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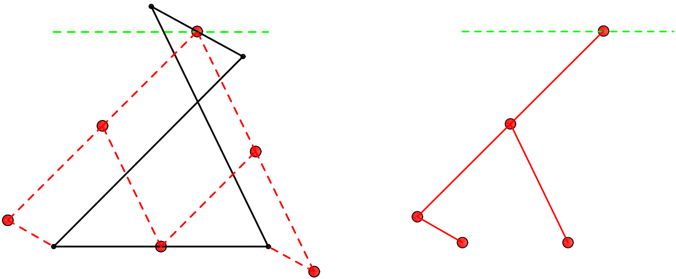

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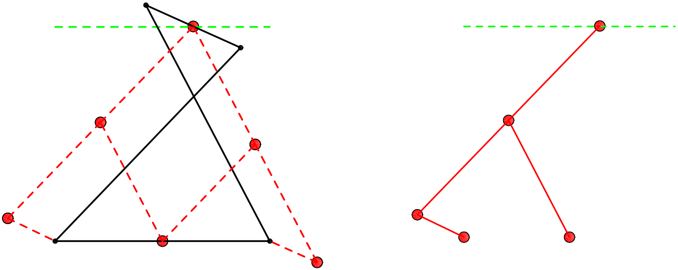

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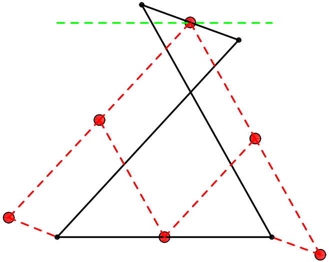

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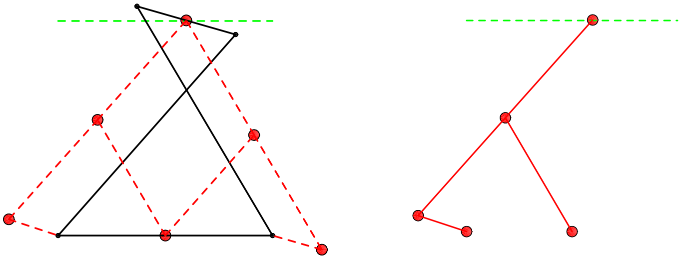

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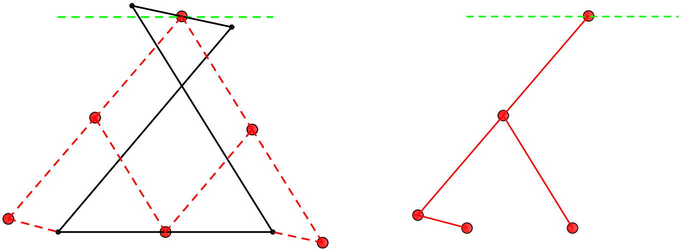

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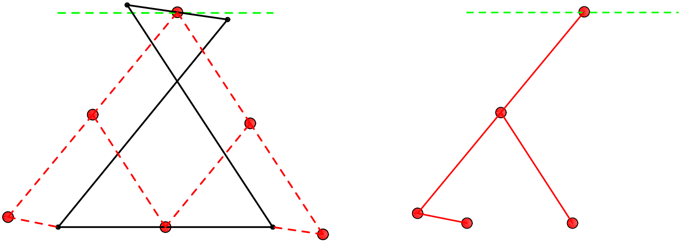

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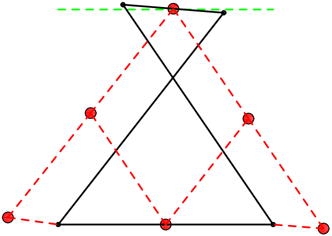

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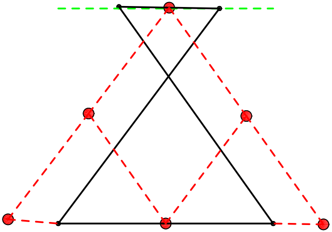

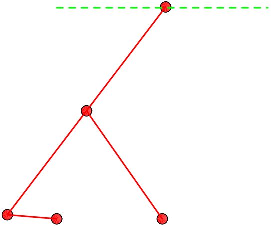

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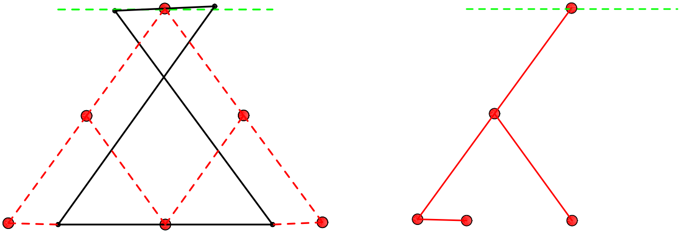

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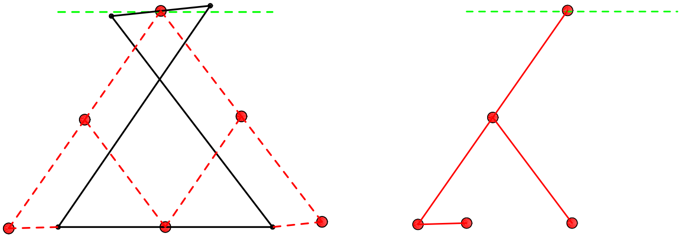

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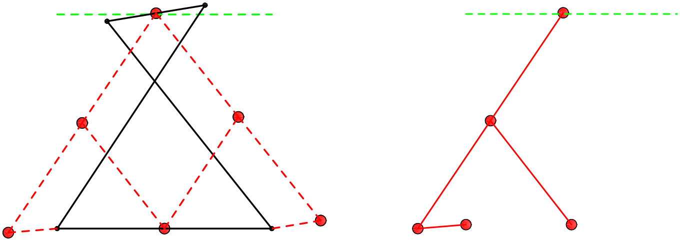

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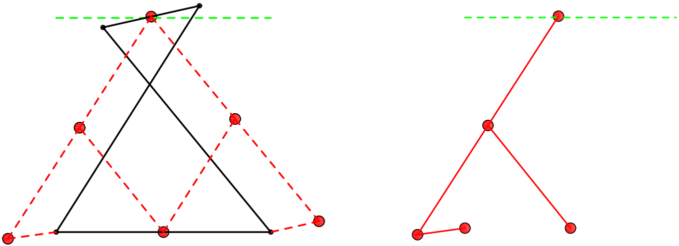

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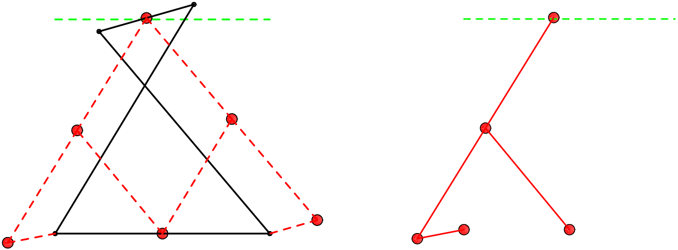

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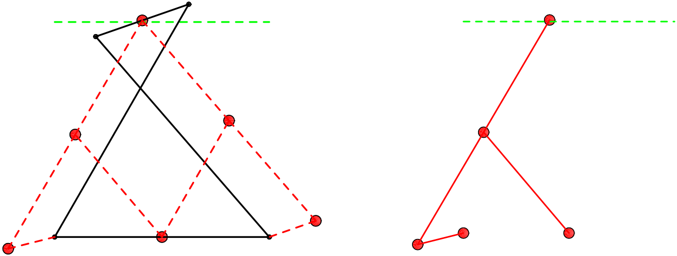

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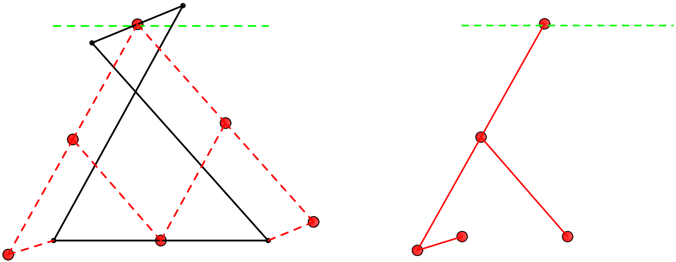

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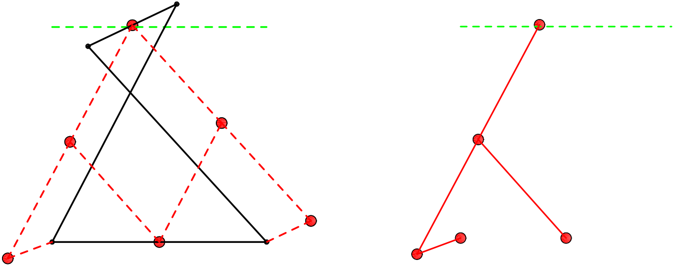

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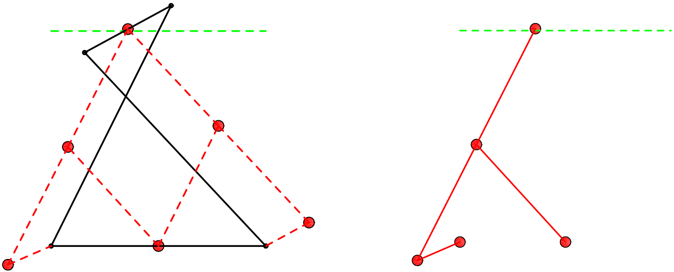

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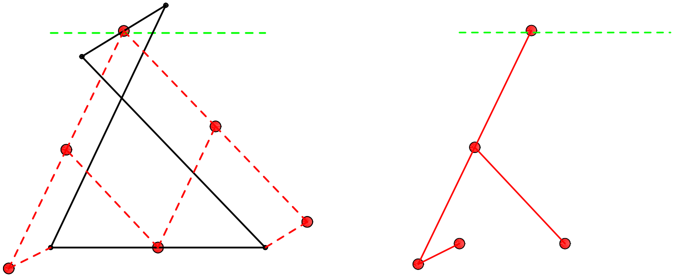

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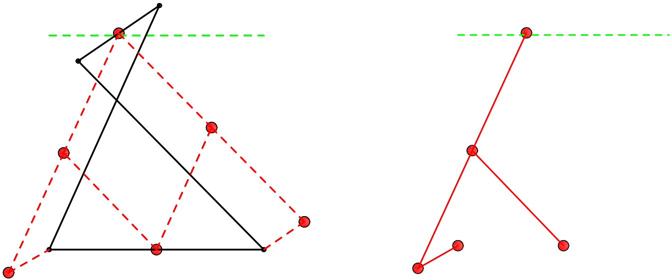

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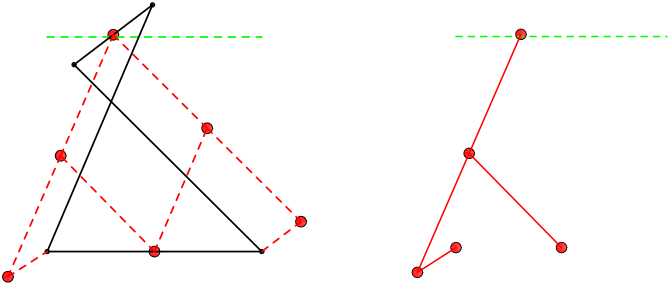

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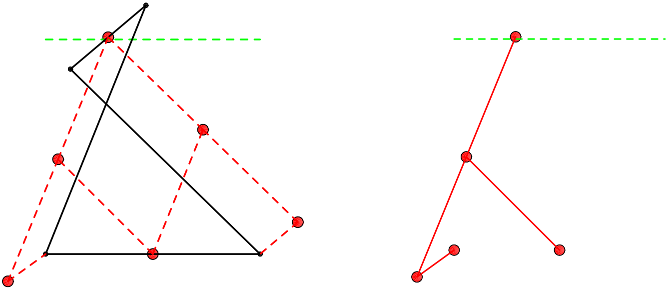

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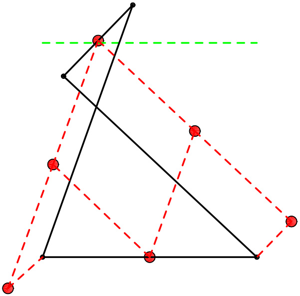

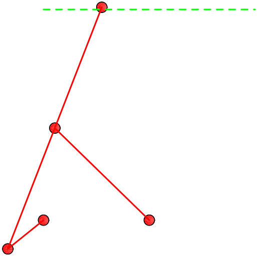

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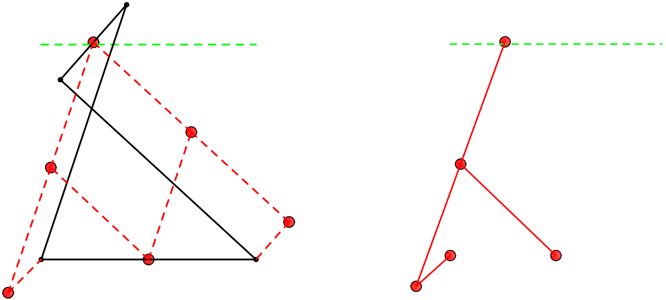

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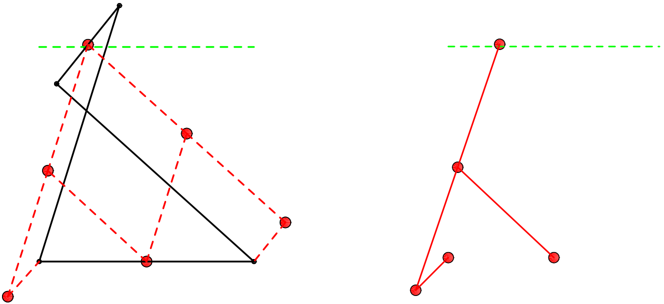

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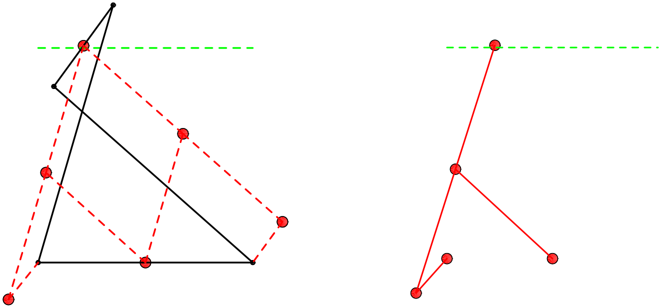

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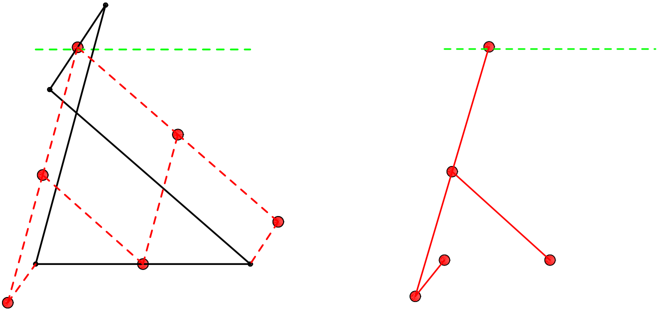

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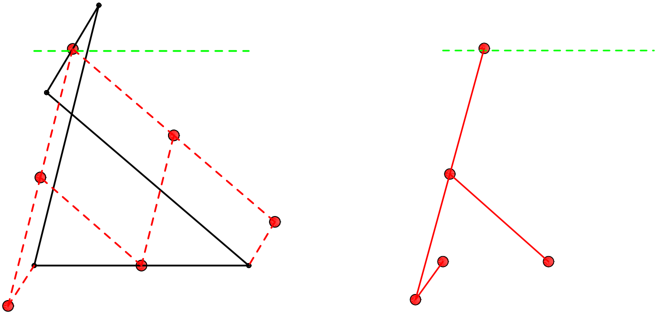

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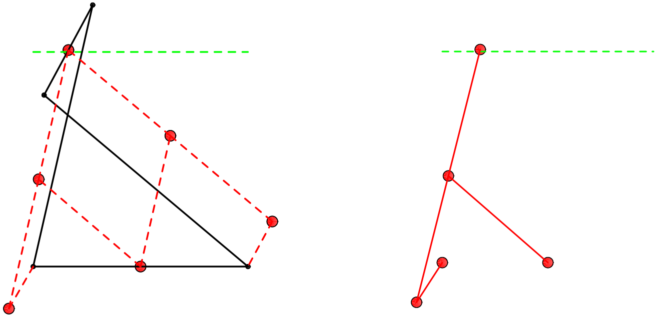

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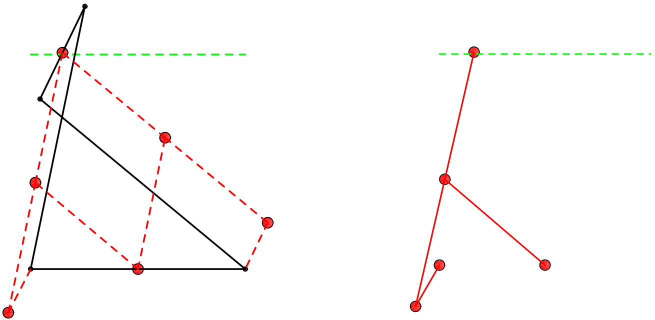

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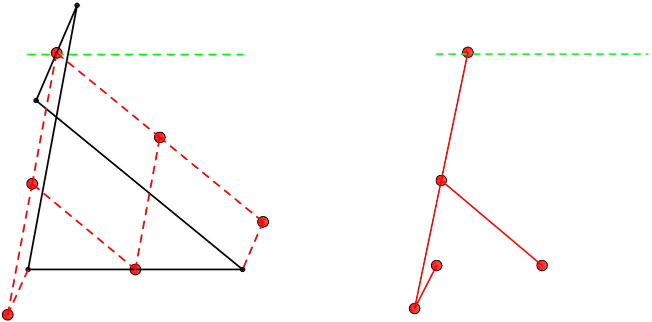

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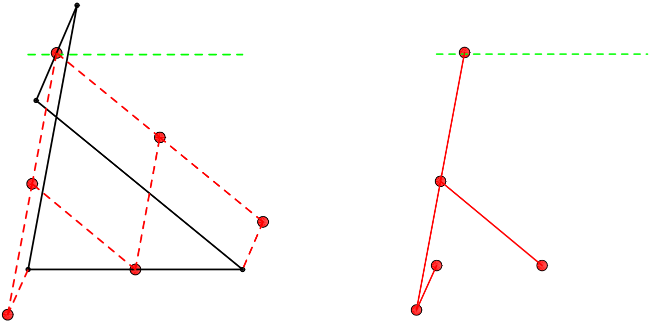

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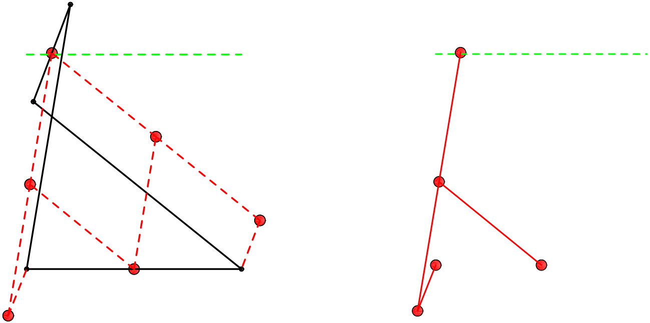

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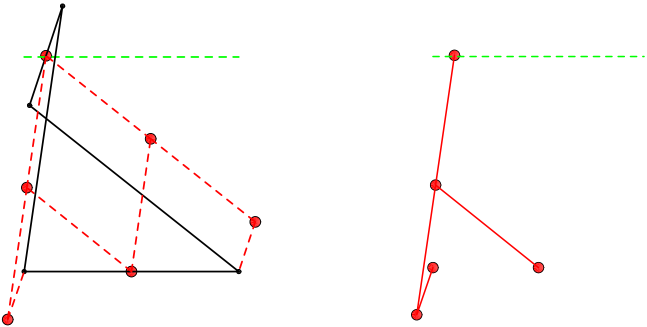

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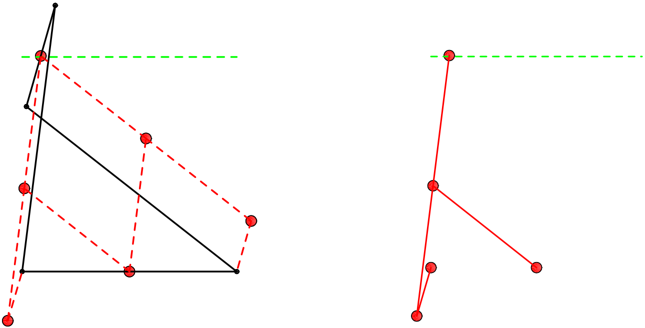

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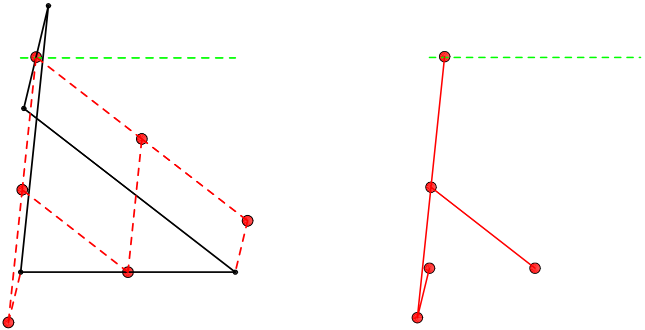

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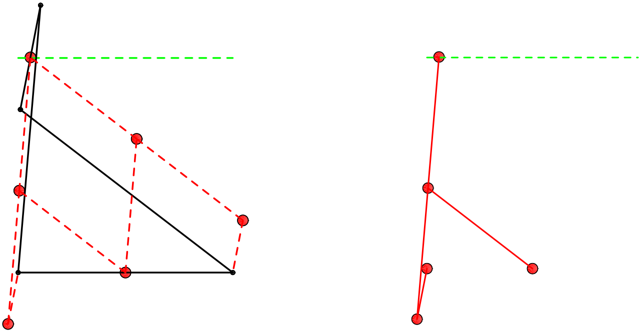

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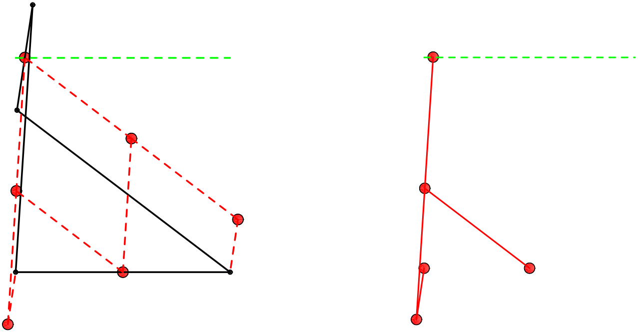

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designing a 4-bar mechanism

A 4-bar mechanism consists of 4 rigid bars,
attached to each other by joints.

Chebyshev's straight line mechanism translates horizontal motion (at the top) into circular motion (at the bottom).

Designing a 4-bar is an interpolation problem:
given coordinates of points on coupler curve (at the top),
find lengths of bars so coupler curve passes through.

A point $(a, b) \in \mathbb{R}^2$ is mapped to $z = a + ib$, $i = \sqrt{-1}$.
Then $(z, \bar{z}) = (a + ib, a - ib) \in \mathbb{C}^2$ are *isotropic coordinates*.

Rotation around $(0, 0)$ through angle θ is multiplication by $e^{i\theta}$. The inverse rotation is $e^{-i\theta}$. Abbreviating a rotation by $\Theta = e^{i\theta}$, then its inverse $\Theta^{-1} = \bar{\Theta}$ and $\Theta\bar{\Theta} = 1$.

the loop equations

Let $A = (a, \bar{a})$ and $B = (b, \bar{b})$ be the fixed base points.

Unknown are $(x, \bar{x})$ and $(y, \bar{y})$, coordinates of the other two points in the 4-bar linkage. For given precision points $(p_j, \bar{p}_j)$, assuming $\theta_0 = 1$, for $j = 1, 2, 3, 4$:

$$\left\{ \begin{array}{l} (p_j + x\theta_j + a)(\bar{p}_j + \bar{x}\bar{\theta}_j + \bar{a}) = (p_0 + x + a)(\bar{p}_0 + \bar{x} + \bar{a}) \\ (p_j + y\theta_j + b)(\bar{p}_j + \bar{y}\bar{\theta}_j + \bar{b}) = (p_0 + y + b)(\bar{p}_0 + \bar{y} + \bar{b}) \end{array} \right.$$

For unknown angles θ_j, associated to $(p_j, \bar{p}_j)$, five precision points determine the linkage uniquely.

Adding $\theta_j \bar{\theta}_j = 1$ gives 12 equations in 12 unknowns:
$(x, \bar{x})$, $(y, \bar{y})$, and $(\theta_j, \bar{\theta}_j)$, for $j = 1, 2, 3, 4$.

reducing to 4 equations

Using $\theta_j \bar{\theta}_j = 1$, we eliminate $\bar{\theta}_j$ via $\bar{\theta}_j = \theta_j^{-1}$, $j = 1, 2, 3, 4$.
Then we view the 8 equations as 4 pairs of linear equations in θ_j and θ_j^{-1}. Applying Cramer's rule to

$$\left\{ \begin{array}{r} \alpha_1 \theta + \alpha_2 \theta^{-1} + \alpha_3 = 0 \\ \beta_1 \theta + \beta_2 \theta^{-1} + \beta_3 = 0 \end{array} \right.$$

eliminates θ and θ^{-1} from the system.
We have solutions only if $\delta_1 \delta_2 + \delta_3^2 = 0$ where δ_1, δ_2, δ_3 are

$$\det \left(\begin{array}{cc} \alpha_2 & \alpha_3 \\ \beta_2 & \beta_3 \end{array} \right), \det \left(\begin{array}{cc} \alpha_1 & \alpha_3 \\ \beta_1 & \beta_3 \end{array} \right), \det \left(\begin{array}{cc} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{array} \right).$$

The 12 dimensional system then reduces
to a system of 4 equations in 4 unknowns.
All equations in the system share the same support.

Kushnirenko’s Theorem

1. Binomial Systems
   - unimodular coordinate transformations
2. Design of 4-bar Mechanisms
   - Chebyshev’s straight line mechanism
3. Regular Triangulations
   - support sets span Newton polytopes
   - the theorem of Kushnirenko
4. Polyhedral Algorithms
   - the Hermite normal form
   - placing points into a regular triangulation

Newton polytopes

The support A of a polynomial f is a finite set of exponent vectors which models the sparse structure of f:

$$f(\mathbf{x}) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad c_{\mathbf{a}} \in \mathbb{C}^*, \quad \mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}.$$

The convex hull of A is the Newton polytope of f,
denoted by $Q = \mathrm{conv}(A)$.

Mostly we work in the plane ($n = 2$) and stick to polygons.

Today we consider systems $f(\mathbf{x}) = \mathbf{0}$ where the equations all share the same support A, or the same Newton polytope Q.

For generic choices of the coefficients, the monomials whose exponent vector is not a vertex do not have an influence on the volume of Q and may be omitted.

regular triangulations

With a triangulation Δ, we compute the volume:

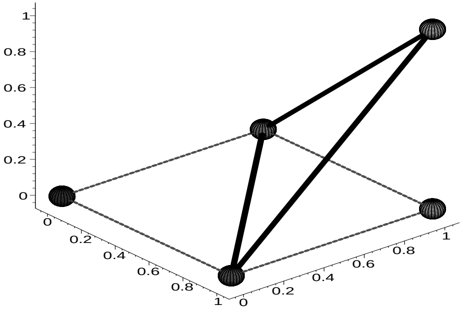
$$\mathrm{vol}(Q) = \sum_{S \in \Delta} \mathrm{vol}(S) \quad S = \mathrm{conv}(C), \ \#C = n + 1.$$

The cells C in a triangulation span simplices S.
For Newton polytopes, the unit simplex has volume 1.
Our triangulations are induced by a lifting function ω:

$$\omega : \mathbb{Z}^n \rightarrow \mathbb{Z} : \mathbf{a} \mapsto \omega(\mathbf{a}),$$

which embeds the support A into $\mathbb{Z}^{n+1}$, $\widehat{A} = \omega(A)$. Accordingly: $\widehat{Q} = \mathrm{conv}(\widehat{A})$. A triangulation is *regular* if there is a lifting function so there is a 1-to-1 mapping of the facets of the lower hull of the lifted point configuration and the simplices in the triangulation.

lifting and homotopy



The heights of the lifted polytope are powers of t in

$$\widehat{g}(\mathbf{x},t) = \left\{ \begin{array}{l} c_{1,11}x_1x_2t^1 + c_{1,10}x_1t^0 + c_{1,01}x_2t^0 + c_{1,00}t^0 = 0 \\ c_{2,11}x_1x_2t^1 + c_{2,10}x_1t^0 + c_{2,01}x_2t^0 + c_{2,00}t^0 = 0. \end{array} \right.$$

At $t = 0$ we find one solution, what about the other?

polyhedral homotopy

To construct the homotopy that starts at system supported at the other cell of the triangulation, we look at the inner normal of the lifted cell, i.e.: $\mathbf{v} = (-1, -1, +1)$.

This inner normal defines the change of coordinates $x_1 = y_1 t^{-1}$ and $x_2 = y_2 t^{-1}$.

After multiplication by t, this change of coordinates yields

$$\widehat{g}(\mathbf{y}, t) = \left\{ \begin{array}{l} c_{1,11} y_1 y_2 t^0 + c_{1,10} y_1 t^0 + c_{1,01} y_2 t^0 + c_{1,00} t^1 = 0 \\ c_{2,11} y_1 y_2 t^0 + c_{2,10} y_1 t^0 + c_{2,01} y_2 t^0 + c_{2,00} t^1 = 0. \end{array} \right.$$

At $\widehat{g}(\mathbf{y}, t = 0) = \mathbf{0}$ there is another solution which contributes one path.

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Puiseux Series

Theorem (the theorem of Puiseux)

Let $f(x_1, x_2) \in \mathbb{C}(x_2)[x_1]$: f is a polynomial in the variable x_1 and its coefficients are fractional power series in x_2.
The polynomial f has as many series solutions as the degree of f.
Every series solution has the following form:

$$\left\{ \begin{array}{l} x_1 = t^u \\ x_2 = ct^v(1 + O(t)), \quad c \in \mathbb{C}^* \end{array} \right.$$

where (u, v) is an inner normal to an edge of the lower hull of the Newton polygon of f.

Using regular triangulations of Newton polytopes,
we can compute Puiseux series for space curves as well.

Kushnirenko's theorem

Theorem (Kushnirenko (1976))

Consider the system $f(\mathbf{x}) = \mathbf{0}$ *and let* A *be the support of every polynomial in* f*. Then the number of isolated solutions of* $f(\mathbf{x}) = \mathbf{0}$ *in* $(\mathbb{C}^*)^n$ *cannot exceed the volume of the polytope spanned by* A*.*

Polyhedral homotopies provide a constructive proof:

1. Any regular triangulation defines a polyhedral homotopy with exactly as many paths as the volume.
2. There are no more solutions than the volume of the Newton polytope, following a refined concept of ∞.

Kushnirenko’s Theorem

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the Hermite normal form

We make zeroes in a matrix via unimodular transformations:

$$\left[\begin{array}{rr} k & \ell \\ -\frac{b}{d} & \frac{a}{d} \end{array}\right] \left[\begin{array}{c} a \\ b \end{array}\right] = \left[\begin{array}{c} d \\ 0 \end{array}\right]$$

where

$$\gcd(a,b) = ka + \ell b = d.$$

The product of unimodular matrices is again unimodular.
Let M_{ij} be the unimodular matrix to make the (i,j)-th element of a matrix zero, then (without pivoting):

$$M = M_{nn-1} \cdots M_{n2} \cdots M_{32} M_{n1} \cdots M_{31} M_{21}, \quad MA = U$$

where U is upper triangular, the Hermite normal form of A.

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placing points

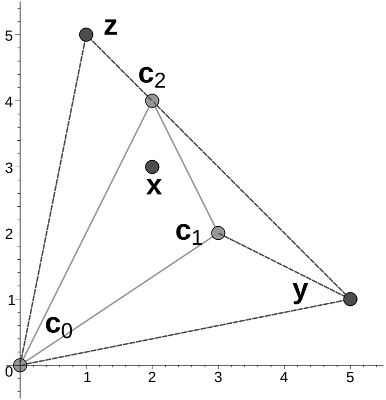
We can compute a regular triangulation, placing points:

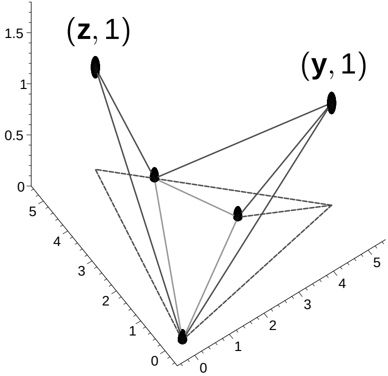
0. $n+1$ linearly independent points span the first cell.
1. For all remaining points:
   1. check if the point is inside the interior of a cell;
   2. if outside, then apply pivoting to find new cells.

Let $[\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2]$, with $\mathbf{c}_0 = (0,0)$, $\mathbf{c}_1 = (3,2)$, and $\mathbf{c}_2 = (2,4)$.

point	barycentric decomposition	pivoting
$\mathbf{x} = (2,3)$:	$\mathbf{x} = +\frac{1}{8}\mathbf{c}_0 + \frac{1}{4}\mathbf{c}_1 + \frac{5}{8}\mathbf{c}_2$	no new simplex
$\mathbf{y} = (5,1)$:	$\mathbf{y} = -\frac{1}{3}\mathbf{c}_0 + \frac{9}{4}\mathbf{c}_1 - \frac{7}{8}\mathbf{c}_2$	$[\mathbf{y}, \mathbf{c}_1, \mathbf{c}_2][\mathbf{c}_0, \mathbf{c}_1, \mathbf{y}]$
$\mathbf{z} = (1,5)$:	$\mathbf{z} = +\frac{1}{8}\mathbf{c}_0 - \frac{3}{4}\mathbf{c}_1 + \frac{13}{8}\mathbf{c}_2$	$[\mathbf{c}_0, \mathbf{z}, \mathbf{c}_2]$

pivoting points





The construction on the right shows how the triangulation can be obtained as the lower hull of **y** and **z** lifted at height one, with $[\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2]$ sitting at level zero.

Summary + Exercises

We stated Kushnirenko's theorem relating the number of solutions to the volume of the Newton polytope.

Exercises:

1. Consider the binomial system $\mathbf{x}^A = \mathbf{c}$ with

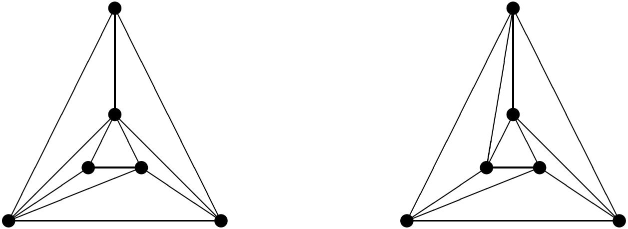
$$A = \left[\begin{array}{ccc} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{array} \right] \quad \text{and} \quad \mathbf{c} = [1 \;\; 1 \;\; 1].$$

   Solve this system. How many solutions do you find? Compare this number with the lowest Bézout bound.

2. Make a Maple worksheet or Sage notebook to generate the system defined by the loop equations. Perform the elimination of the θ variables using Cramer's rule. Solve the polynomial system and verify that for random choices of the parameters there are indeed as many solutions as the volume of the Newton polytope.

more exercises

3. Consider $A = \{(0,0),(8,0),(4,8),(3,2),(4,4),(5,2)\}$ and two triangulations below:



Show that the triangulation on the left is regular, while the one on the right is not.

and more exercises

4. Instead of defining the support of a polynomial as the collection of exponents for which the corresponding coefficient is different from zero, we can first prescribe the support A and then consider all polynomials whose support is a subset of A.
    1. What is the prescribed support and Newton polytope of the theorem of Bézout? Give an example in 2 variables and interpret Bézout's theorem by the application of Kushnirenko's theorem.
    2. Do the same for the multihomogeneous version of Bézout.
5. Give an example of a class of systems of two polynomial equations in two variables with shared support so that the area of the Newton polygon is much less than the product of the degrees or the 2-homogeneous Bézout bound.
   What seems to be typical for such systems?

one last exercise

6. Consider the homotopy

$$h(x_1, x_2, t) = \begin{cases} x_1x_2 + x_1 - x_2 + t = 0 \\ x_1x_2 - 2x_1 + x_2 + t = 0 \end{cases}$$

    1. The series solution up to fourth order is

    $$\left(\frac{1}{3} - 2t - 12t^2 - 144t^3 + O(t^4), \right.$$

    $$\left. \frac{1}{2} - 3t - 18t^2 - 216t^3 + O(t^4) \right).$$

    Develop the series solution further up to $O(t^8)$.

    2. Compute the other series solution for this system, using a proper coordinate transformation.