

Quotient Rings

In this note we consider again ideals, but here we do not start from polynomials, but from a finite set of points. The application in statistics and the pseudo code of the Buchberger-Möller algorithm are from [11].

1 Defining Ideals and the Shape Lemma

Consider a set $V \subset \mathbb{C}^n$. The defining ideal of V is the set of all polynomials which vanish on V , or formally:

$$I(V) = \{ f \in \mathbb{C}[\mathbf{x}] \mid f(\mathbf{z}) = \mathbf{0}, \forall \mathbf{z} \in V \}. \quad (1)$$

If V contains only one point $\mathbf{z} = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$, $V = \{\mathbf{z}\}$, then $I(V) = \langle x_1 - z_1, x_2 - z_2, \dots, x_n - z_n \rangle$.

For any ideal I , the quotient ring is

$$\mathbb{C}[\mathbf{x}]/I = \{ r \in \mathbb{C}[\mathbf{x}] \mid r \text{ is remainder of the division of } f \text{ by } I, \forall f \in \mathbb{C}[\mathbf{x}] \}. \quad (2)$$

For $I = I(\{\mathbf{z}\})$ we see that the remainder after division by the polynomials $x_i - z_i$, $i = 1, 2, \dots, n$, will always be a constant. Thus we may identify $\mathbb{C}[\mathbf{x}]/I(\{\mathbf{z}\})$ with \mathbb{C} .

Finding a description for $I(V)$ is closely related to interpolating. Consider $V = \{\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_k\}$ a collection of k distinct points in \mathbb{C}^n . A polynomial $s_{\mathbf{z}_i}(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]$ is a separator for the point \mathbf{z}_i if $s_{\mathbf{z}_i}(\mathbf{z}_j) = 1$ if $j = i$ and 0 if $j \neq i$. We denote by $z_{i,j}$ the j th coordinate of \mathbf{z}_i . If all points in V have a different first coordinate $z_{i,1}$, then we can use the Lagrange polynomials in the first coordinate to separate the points from each other. In particular, the i th Lagrange polynomial is then

$$L_i(\mathbf{x}) = \frac{(x_1 - z_{1,1}) \cdots (x_1 - z_{i-1,1})(x_1 - z_{i+1,1}) \cdots (x_1 - z_{k,1})}{(z_{i,1} - z_{1,1}) \cdots (z_{i,1} - z_{i-1,1})(z_{i,1} - z_{i+1,1}) \cdots (z_{i,1} - z_{k,1})}. \quad (3)$$

Given a finite set of values v_i for each point $\mathbf{z}_i \in V$, the separators $s_{\mathbf{z}_i}$ are useful to define an interpolating polynomial $p \in \mathbb{C}[\mathbf{x}]$ such that $p(\mathbf{z}_i) = v_i$, for $i = 1, 2, \dots, k$. It is straightforward to verify that

$$p(\mathbf{x}) = \sum_{i=1}^k v_i s_{\mathbf{z}_i}(\mathbf{x}) \text{ satisfies the } k \text{ interpolation conditions.}$$

As a special case of the interpolating polynomials, consider the polynomials

$$p_j(\mathbf{x}) = x_j - \sum_{i=1}^k z_{i,j} s_{\mathbf{z}_i}(\mathbf{x}), \quad j = 1, 2, \dots, n, \quad (4)$$

where the values v_i are simply the coordinates of the points. Evaluating p_j at \mathbf{z}_ℓ , we see that

$$p_j(\mathbf{z}_\ell) = z_{\ell,j} - \sum_{i=1}^k z_{i,j} s_{\mathbf{z}_i}(\mathbf{z}_\ell) = z_{\ell,j} - z_{\ell,j} s_{\mathbf{z}_\ell}(\mathbf{z}_\ell) = 0. \quad (5)$$

If all points have a distinct first coordinate, then we can again use the Lagrange polynomials as separators and construct a lexicographic Gröbner basis for $I(V)$ as follows:

$$g = \left\{ x_n - \sum_{i=1}^k z_{i,n} L_i(\mathbf{x}), \dots, x_2 - \sum_{i=1}^k z_{i,2} L_i(\mathbf{x}), (x_1 - z_{1,1})(x_1 - z_{2,1}) \cdots (x_1 - z_{k,1}) \right\}. \quad (6)$$

By its shape, we see that the solution set of this set of equations equals V . This is known as the Shape Lemma [7]. Applying the division algorithm on any polynomial $f \in \mathbb{C}[\mathbf{x}]$ with respect to $\langle g \rangle$ will only leave remainders of the form $r = r_0 + r_1 x_1 + r_2 x_1^2 + \cdots + r_{k-1} x_1^{k-1}$. Thus $\dim(\mathbb{C}[\mathbf{x}]/\langle g \rangle) = k$.

If some of the points in V have the same first coordinate, then we may apply a random linear coordinate change to ensure that all points have a different first coordinate. Geometrically, Lagrange interpolation can thus always be applied. Algebraically, a random coordinate change might not be so desirable.

2 Design of Experiments

Suppose a firm wants to test the reaction of consumer on a new product, determined by several features, such as color, shape, weight, material, and price. Assuming three different choices for each features, there would be $3^5 = 243$ many possibilities. Even if the firm could build that many prototypes, it would be difficult to find consumers who would be willing to evaluate and rate that many models. The set of all possible choices is called a full design, denoted by \mathcal{D} .

The problem is then to determine a suitable subset $\mathcal{F} \subset \mathcal{D}$ from which we can construct a good model to rate the product. By a model we mean a polynomial in the features which returns a quality rating of the product. More precisely, if the five features are color (C), shape (S), weight (W), material (M), and price (P), then a model is a polynomial $f \in \mathbb{R}[C, S, W, M, P]$ which return a rating (e.g. on a scale of 0 to 10) about the product, based on its chosen features.

For full designs – considering all possible combinations of the features – we may construct the model using plain linear algebra, i.e.: by just solving a linear system. For example, if we have two choices for three features, represented by the variables x_1 , x_2 , and x_3 , then there are 2^3 coefficients in the model f to determine where $f(x_1, x_2, x_3) = c_0 + c_1x_1 + c_2x_2 + c_3x_3 + c_4x_1x_2 + c_5x_1x_3 + c_6x_2x_3 + c_7x_1x_2x_3$.

For factorial designs, we have to construct a model f from a given set of points, with sampled values for the features with corresponding customer ratings.

Algebraic statistics is a new field, emerged in the mid nineties, see [10] and [9] for introductions.

Following [11], we list three problems:

1. Let $\mathcal{F} := \{\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_s\} \subset \mathcal{D}$. What are the complete linear models which can be identified by \mathcal{F} ?
2. Let the polynomial $f(\mathbf{x})$ be a complete linear polynomial model, where the support of f is a subset of the normal set of \mathcal{D} . What are the minimal fractions which identify the model?
3. What are the fractions \mathcal{F} of \mathcal{D} , with $\#\mathcal{F} < \#\mathcal{D}$ which identify the highest (or lowest) number of complete linear polynomial models?

We defer to [11] for examples and solutions to these problems.

3 The Quotient Algebra

Via interpolation in a finite set V of k distinct points we derived descriptions for its defining ideal $I(V)$. One important observation we made was that the quotient ring is a k -dimensional vector space. A ring that is also a vector space is called an algebra.

Consider the Gröbner basis (example taken from [5]), with respect to the total degree order ($x > y$):

$$g = \{ 2x^2 + 3xy + y^2 - 3x - 3y, xy^2 - x, y^3 - y \}. \quad (7)$$

The initial ideal is generated by the leading monomials of g : $\langle x^2, xy^2, y^3 \rangle$. The only monomials not in the ideal are

$$O = \{1, x, y, xy, y^2\} \Rightarrow \forall p \in \mathbb{C}[x, y] : p \rightarrow_g r = c_1 + c_2x + c_3y + c_4xy + c_5y^2 \quad (8)$$

for some coefficients c_i , $i = 1, 2, \dots, 5$. In Maple we type `Groebner[NormalForm](x^2*y, g, tdeg(x, y))`; to compute the remainder of the division by g . Turning the set O into a basis vector B using \rightarrow_g on xB , we define an eigenvalue problem $xB = M_x B$:

$$x \begin{bmatrix} 1 \\ x \\ y \\ xy \\ y^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{3}{2} & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{3}{2} & -\frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ xy \\ y^2 \end{bmatrix} \quad (9)$$

where the matrix M_x defines the multiplication by x , modulo the ideal. The multiplication by y defines M_y . The eigenvalues of M_x and M_y give respectively the x and y -coordinates of the solutions. So the multiplication matrices generalize the familiar companion matrices for polynomials in one variable. This gives rise to what is called “the central theorem of polynomial systems solving” in [12], originating in [2].

The original motivation for the development of Gröbner bases was to find algorithms for computing a basis for the quotient ring [3]. If the set of monomials not in the ideal is a finite set (as the O of above), then we can compute a multiplication table of all monomials in O to represent the computations in this finite dimensional algebra.

Observe that we have given constructive proofs for the following theorem:

Theorem 3.1 Consider $f(\mathbf{x}) = \mathbf{0}$ a polynomial system with $V = f^{-1}(\mathbf{0}) \subset \mathbb{C}^n$. Assume that all solutions have multiplicity one. Then:

$$\#V < \infty \iff \dim(\mathbb{C}[\mathbf{x}]/\langle f \rangle) = \#V. \quad (10)$$

In the \Rightarrow direction, we have given a finite collection of points in V and with the Shape lemma and Lagrange polynomials we can compute a lexicographic Gröbner basis for which the quotient ring will have a finite basis. For the \Leftarrow direction, we define the multiplication matrices to compute the coordinates of all points in V .

4 The Central Theorem

We consider zero dimensional ideals $I = \langle f \rangle$, defined by a polynomial system $f(\mathbf{x}) = \mathbf{0}$, with approximate coefficients. Consider the system (from [8]):

$$f(\mathbf{x}) = \begin{cases} x_1 + x_2 + x_3 - 1 = 0 \\ \frac{1}{5}x_1^3 + \frac{1}{2}x_2^2 - x_3 + \frac{1}{2}x_3^2 + \frac{1}{2} = 0 \\ x_1 + x_2 + \frac{1}{2}x_3^2 - \frac{1}{2} = 0. \end{cases} \quad (11)$$

We can solve it with Maple (available since Maple 7, see [4]):

```
[> gb := Groebner[gbasis](P,tdeg(x[1],x[2],x[3]));
[> ns,rv := Groebner[SetBasis](gb,tdeg(x[1],x[2],x[3]));
```

The last command returns the normal set as a list in `ns` and as a table in `rv`: $[1, x_3, x_2, x_2x_3, x_2^2, x_2^2x_3]$. The size of the normal set tells us that the system has six solutions. We continue the Maple session with the computation of the multiplication matrices and their eigenvalues:

```
[> Mx[1] := Groebner[MulMatrix](x[1],ns,rv,gb,tdeg(x[1],x[2],x[3]));
[> v1 := LinearAlgebra[Eigenvalues](Mx[1]);
```

The eigenvalues of the multiplication matrix M_{x_1} are $[-\frac{5}{2}, -\frac{5}{2}, 0, 0, 0, 0]$. Computing the eigenvalues of M_{x_2} and M_{x_3} gives $[-\frac{5}{2}, -\frac{5}{2}, 0, 0, 0, 0]$ and $[1, 1, 1, 1, 1, 1]$, so we can see that there are two distinct solutions: $(-\frac{5}{2}, -\frac{5}{2}, 1)$ and $(0, 0, 1)$, respectively occurring with multiplicities two and four. The Jordan canonical form can be computed as

```
[> J := LinearAlgebra[JordanForm](Mx[1]);
```

and returns

$$J = \left[\begin{array}{cc|cccc} -\frac{5}{2} & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{5}{2} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]. \quad (12)$$

In the display of J we marked the different elementary Jordan blocks with each eigenvalue.

We paraphrase what is stated as a simple version of the central theorem [12]:

Theorem 4.1 *Let I be a zero dimensional ideal with $\#V(I) = k$, $V(I) \subset \mathbb{C}^n$. Then there is a commuting family of multiplication matrices M_{x_i} , each of dimension k , for $i = 1, 2, \dots, n$. The multiplication matrices have a joint invariant subspace. Each joint eigenvector corresponds to one solution.*

This so-called central theorem establishes the link between zero dimensional polynomial ideals and eigenvalue problems. Because eigenvalue problems are so well studied in numerical linear algebra, and widely used in scientific computing, the problem of solving a zero dimensional polynomial system may be translated in a task of numerical linear algebra.

The Jordan Canonical Form (abbreviated by JCF) plays a central role in linear algebra. Consider the eigenvalue problem $A\mathbf{x} = \lambda\mathbf{x}$, for A a real or complex n -by- n matrix.

If all eigenvalues are distinct, then the corresponding eigenvectors form a basis for n -space. In particular, storing the eigenvectors in the columns of a matrix V , we obtain the decomposition $AV = V\Lambda$, where Λ is the diagonal matrix with the eigenvalues on its diagonal. Because V is invertible, we may write $A = V\Lambda V^{-1}$.

If an eigenvalue λ occur multiple times, say m times, then there exist k elementary Jordan blocks for λ , each of size m_i , with $m = m_1 + m_2 + \dots + m_k$. An elementary Jordan block has λ on its diagonal, ones on its upper diagonal and zeroes everywhere else. The number k of elementary Jordan blocks is the geometric multiplicity while m is the algebraic multiplicity of λ . A matrix is said to be nonderogatory if it has only eigenvalues of geometric multiplicity equal to one.

Even with multiple eigenvalues, if we sort the distinct eigenvalues and the sizes of the elementary Jordan blocks for each multiple eigenvalue, then there is a unique decomposition of the matrix A and the factorization of A into ZJZ^{-1} , where J is a block diagonal matrix with on its diagonal the elementary Jordan blocks, and for Z a nonsingular matrix, is called the Jordan canonical form of A .

Let $A \in \mathbb{C}^{n \times n}$. The Jordan canonical form of A is a block diagonal matrix J for which there exists a nonsingular matrix P so $AP = PJ$ or $J = P^{-1}AP$, where

$$J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_l \end{bmatrix} \quad \text{and} \quad J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix}, i = 1, 2, \dots, l. \quad (13)$$

The matrices J_k are elementary Jordan blocks and $\text{eig}(A) = \{\lambda_1, \lambda_2, \dots, \lambda_l\}$ are the l distinct eigenvalues of A . For every eigenvalue λ , there are k elementary Jordan blocks of orders $m_1 \geq m_2 \geq \dots \geq m_k$. The geometric multiplicity of λ equals then k while $m = m_1 + m_2 + \dots + m_k$ is the algebraic multiplicity of λ . For example, for $n = 3$, we may have three different configurations of m and k :

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \quad \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \quad \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}, \quad (14)$$

$m = 3, k = 3$ $m = 3, k = 2$ $m = 3, k = 1$.

In general, for $\lambda \in \text{eig}(A)$ we decompose A into $A[X|Y] = [X|Y] \left([J_\lambda | \tilde{A}] \right)$, where $\lambda \notin \text{eig}(\tilde{A})$. The matrix X contains in its m columns an invariant subspace associated with λ , i.e.: $\mathbf{v} \in X: A\mathbf{v} \in X$.

Up to the order of the elementary Jordan blocks, the Jordan canonical decomposition is unique, therefore it is justified to speak of ‘the’ Jordan form of a matrix.

5 The Buchberger-Möller Algorithm

In this section we give an algorithm to compute a Gröbner basis for the ideal $I(V)$ of a finite collection of points V .

A set of monomials O is called a *normal set* when for all $\mathbf{x}^a \in O$: if \mathbf{x}^b divides x^a , then $\mathbf{x}^b \in O$. For all generators \mathbf{x}^a of $\langle \text{LT}(I(V)) \rangle$ (where LT returns the leading term of a polynomial with respect to some term order), we have $\mathbf{x}^a = x_i \mathbf{x}^b$ for some i and $\mathbf{x}^b \in O$. Note: $\dim(\mathbb{C}[\mathbf{x}]/I(V)) = \#V = \#O$ and the normal set O is a basis for the quotient ring $\mathbb{C}[\mathbf{x}]/I(V)$.

The normal set refers to the normal form algorithm: with a Gröbner basis, the division algorithm gives a unique remainder that can be expressed as a linear combination of the monomials in the normal set. If the generators of the ideal have a finite set of common zeroes, then the normal set is finite. The elements in the normal set define the border of the staircase. Starting with 1, the algorithm below explores the extension of the region under the staircase.

Algorithm 5.1 The Buchberger-Möller Algorithm

Input: $V = \{\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_k\}$ and a term order $<$.
 Output: g is a Gröbner basis of $I(V)$ with respect to $<$;
 O is a normal set, $\#O = k$;
 S is a set of separators for the points in V .

```

 $g := \emptyset$ ;  $O := \emptyset$ ;  $S := \emptyset$ ;  $r := 0$ ;  $L := \{1\}$ ;
while  $L \neq \emptyset$  do
   $T := \min_{<}(L)$ ;  $L := L \setminus T$ ;
   $f := T - \sum_{i=1}^k T(\mathbf{z}_{\pi(i)})s_i$ ;
  if  $f$  vanishes on  $V$  then
     $g := g \cup \{f\}$ ;
     $L := L \setminus \{ \text{multiples of } T \}$ ;
  else
     $O := O \cup T$ ;  $r := r + 1$ ;
     $\pi(r) := \min\{ i \mid f(\mathbf{z}_i) \neq 0 \}$ ;
     $s_r := f/f(\mathbf{z}_{\pi(r)})$ ;
     $S := S \cup \{s_r\}$ ;
    for  $i$  from 1 to  $r - 1$  do
       $s_i := s_i - s_i(\mathbf{z}_{\pi(r)})s_r$ ;
    end for;
     $L := L \cup \{ x_i T \mid i = 1, 2, \dots, n \} \setminus \text{LT}(g)$ ;
  end if;
end while.
```

According to [11], the cost of the Buchberger-Möller algorithm is quadratic in n and cubic in k .

6 using Macaulay 2

An implementation of the Buchberger-Möller algorithm is available in the Macaulay 2 package `Points`, see [6]. An implementation in CoCoA is described in [1]. Below is a session with Macaulay 2 to show several examples.

```

$ M2
Macaulay2, version 1.3.1
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
               PrimaryDecomposition, ReesAlgebra, SchurRings, TangentCone

i1 : M = matrix{{1,2,3},{4,5,6}}

o1 = | 1 2 3 |
     | 4 5 6 |
           2       3
o1 : Matrix ZZ <--- ZZ

i2 : R = QQ[x,y,MonomialOrder=>Lex];

i3 : loadPackage "Points";

i4 : (Q,inG,G) = points(M,R)

o4 = ({1, y, y^2}, ideal (y^3, x), {y^3 - 15y^2 + 74y - 120, x^2 - y + 3})
o4 : Sequence

i5 : M2 = matrix{{1,2,3},{2,2,2}}

o5 = | 1 2 3 |
     | 2 2 2 |
           2       3
o5 : Matrix ZZ <--- ZZ

i6 : (Q2,inG2,G2) = points(M2,R)

o6 = ({1, x, x^2}, ideal (y, x^3), {y^2 - 2, x^3 - 6x^2 + 11x - 6})
o6 : Sequence

i7 : M3 = matrix{{1,1,2},{1,2,2}}

o7 = | 1 1 2 |
     | 1 2 2 |
           2       3
o7 : Matrix ZZ <--- ZZ

i8 : (Q3,inG3,G3) = points(M3,R)

o8 = ({1, y, x}, ideal (y^2, x*y, x^2), {y^2 - 3y + 2, x*y - 2x - y + 2, x^2 - 3x + 2})
o8 : Sequence

```

7 Exercises

1. For two finite sets V and W , show that $V \subset W$ implies $I(V) \supset I(W)$.
2. Use the definition of the Gröbner basis to show that the set of polynomials (6) defined by the Shape lemma is a Gröbner basis for $I(V)$.
3. Consider $V = \{(1/3, 5, 1/2), (1, 2, 1/5), (2, 2, 3), (2, 2, 2)\}$. Observe that the third coordinate is different for each point in V . Use this observation to compute the shape lemma representation as in (6) of V .
4. For the set V as in the previous exercise, take a lexicographic term order and run the Buchberger-Möller algorithm.
5. Download the free computer algebra system CoCoA from <http://cocoa.dima.unige.it/> and use it to solve the previous exercises. Alternatively, you may also use Macaulay 2.
6. Consider the ideal (taken from [5]) $I = \langle xy + z - xz, x^2 - z, 2x^3 - x^2yz - 1 \rangle$ over $\mathbb{Q}[x, y, z]$. Use CoCoA, Maple, Macaulay 2, or Sage for the following tasks:
 - (a) Compute a Gröbner basis with respect to the lexicographic order $x > y > z$. How many points does $V(I)$ have? Give the coordinates for all points in $V(I)$.
 - (b) Use $V(I)$ to bring the ideal in the shape lemma and compare with the lexicographic Gröbner basis.
 - (c) Use the total degree order to compute a Gröbner basis g for I . What is the normal set and corresponding basis vector B for the quotient ring? Apply the division algorithm \rightarrow_g to define M_x . Compute the eigenvalues for M_x and verify that they give the x -coordinates of the points in $V(I)$.

References

- [1] J. Abbott, A. Bigatti, M. Kreuzer, and L. Robbiano. Computing ideals of points. *Journal of Symbolic Computation*, 30(4):341–356, 2000.
- [2] W. Auzinger and H.J. Stetter. An elimination algorithm for the computation of all zeros of a system of multivariate polynomial equations. In *Numerical Mathematics, Proceedings of the International Conference, Singapore*, volume 86 of *International Series of Numerical Mathematics*, pages 11–30. Birkhäuser, 1988.
- [3] B. Buchberger. An algorithmic criterion for the solvability of a system of algebraic equations. In B. Buchberger and F. Winkler, editors, *Gröbner Basis and Applications*, volume 251 of *London Mathematical Society Lecture Notes Series*, pages 535–545. Cambridge University Press, 1998. Translation of ‘Ein algorithmisches Kriterium für die Lösbarkeit eines algebraischen Gleichungssystems’, *Aeq. Math* 4: 374-383, 1970.
- [4] R.M. Corless. Groebner bases and matrix eigenproblems. *SIGSAM Bulletin*, 30(4):26–32, 1996.
- [5] D. Cox, J. Little, and D. O’Shea. *Using Algebraic Geometry*, volume 185 of *Graduate Texts in Mathematics*. Springer-Verlag, 1998.
- [6] W. Just and B. Stigler. Computing gröbner bases of ideals of few points in high dimensions. *ACM Communications in Computer Algebra*, 40(3/4):67–78, 2006.
- [7] M. Kreuzer and L. Robbiano. *Computational Commutative Algebra 1*. Springer-Verlag, 2000.
- [8] T. Ojika, S. Watanabe, and T. Mitsui. Deflation algorithm for the multiple roots of a system of nonlinear equations. *J. Math. Anal. Appl.*, 96:463–479, 1983.

- [9] G. Pistone, E. Riccomagno, and H.P. Wynn. *Algebraic Statistics. Computational Commutative Algebra in Statistics*, volume 89 of *Monographs on Statistics and Applied Probability*. Chapman & Hall/CRC, 2001.
- [10] G. Pistone and H.P. Wynn. Generalised confounding with gröbner bases. *Biometrika*, 83(3):653–666, 1996.
- [11] L. Robbiano. Gröbner bases and statistics. In B. Buchberger and F. Winkler, editors, *Gröbner Basis and Applications*, volume 251 of *London Mathematical Society Lecture Notes Series*, pages 179–204. Cambridge University Press, 1998.
- [12] H.J. Stetter. *Numerical Polynomial Algebra*. SIAM, 2004.