Coefficient-Parameter Homotopies

Polynomial systems arising in practical applications have parameters. Solving the system with generic parameters provides a start system in the homotopy to solve specific instances of the problem. The cover of [3] shows a parallel robot, an important application from mechanical design.

1 Generic Choices and Parameter Homotopies

If we want to solve $f(\mathbf{x}) = \mathbf{0}$ using a start system $g(\mathbf{x}) = \mathbf{0}$ then we introduce an artificial parameter t and form the homotopy

$$h(\mathbf{x}, t) = \gamma(1-t)g(\mathbf{x}) + tf(\mathbf{x}) = 0, \quad \gamma \in \mathbb{C}, \quad t \in [0,1]. \tag{1}$$

Via elimination methods we can compute those γ for which $(\mathbf{x}, t)$ satisfy $h(\mathbf{x}, t) = \mathbf{0}$ and $\det(\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}, t)) = 0$. Since there are only finitely many such *complex* values for γ, we know singular solutions do not occur for all t: $0 \leq t < 1$.

The key word we use is "generic" in the sense of algebraic geometry. A choice for a value of a parameter γ is generic if it does not belong to a certain set defined by discriminant conditions. The use of *complex* values of γ is also very important: the critical values for γ lie in the complex plane. The continuation parameter t lives in the interval $[0, 1]$. Except possibly for $t = 1$, the discriminant conditions for γ will not give values in that interval.

Most polynomial systems arising in practical applications naturally come with parameters $\boldsymbol{\lambda}$. Then we write our system as $f(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{0}$. Suppose we know solutions of $f(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{0}$ for parameters $\boldsymbol{\lambda} = \boldsymbol{\lambda}_0$, then we may define the coefficient-parameter homotopy

$$h(\mathbf{x}, t) = f(\mathbf{x}, (1-t)\boldsymbol{\lambda}_0 + t\boldsymbol{\lambda}_1) = \mathbf{0}, \quad t \in [0,1], \tag{2}$$

to find solutions of $f(\mathbf{x}, \boldsymbol{\lambda}_1) = \mathbf{0}$.

The main distinction between the artificial-parameter homotopy in (1) and the natural-parameter homotopy in (2) mainly lies in the number of paths that must be tracked. The performance of the homotopy (1) mainly depends on how good the start system $g(\mathbf{x}) = \mathbf{0}$ matches the structure of the system which must be solved.

As we will see in the next application, parameters may be nested. If $\boldsymbol{\lambda} \in Q$, $Q \subset \mathbb{C}^m$, then we may consider the sequence $Q \supset Q_1 \supset Q_2 \supset \cdots$ of nested parameter spaces. These parameter spaces may be defined algebraically by polynomial equations, i.e.: by adding extra linear equations cutting down on the dimensions of the parameter space or simply by specializing certain values of the parameters. Besides the relevant of nested parameter spaces, we may intuitively use this construction in an intuitive induction argument to justify the regularity of the solution paths.

In addition to imposing restrictions on the parameter space, we may also restrict the space for the variables $\mathbf{x}$ to belong to $U \subset \mathbb{C}^n$. A typical restriction on $\mathbf{x}$ is that not all coordinates are equal to zero, which translates into the polynomial equation $x_1 x_2 \cdots x_n = 0$ *not* being satisfied. The conditions on $\mathbf{x}$ are called side conditions.

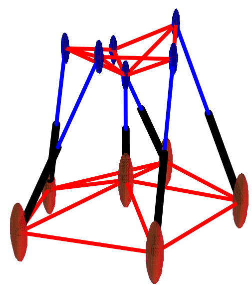
In summary, we consider polynomial systems given as $f(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{0}$, with side conditions imposed on $\mathbf{x} \in U$ and parameters $\boldsymbol{\lambda} \in Q$. Note that while Q is an algebraic set, U is the complement of an algebraic set. Then we define a root count $\mathcal{N}(\boldsymbol{\lambda}, U)$ as

$$\mathcal{N}(\boldsymbol{\lambda}, U) = \# \left\{ \mathbf{x} \in U \mid f(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{0} \text{ and rank} \left(\frac{\partial f}{\partial x}(\mathbf{x}, \boldsymbol{\lambda}) \right) = n \right\}. \tag{3}$$

So the root count $\mathcal{N}(\boldsymbol{\lambda}, U)$ defines the number of regular solutions subject to side conditions of a system for $\boldsymbol{\lambda} \in Q$, where Q is some parameter space. That the definition of $\mathcal{N}(\boldsymbol{\lambda}, U)$ makes sense is an important theorem with many applications.

2 Stewart-Gough Platforms

Our application concerns the so-called forward displacement problem of a platform manipulator, following [4]. A platform manipulator – think about a flight simulator – consists of a moving end plate supported by six extensible legs from a base plate.



Forward Displacement Problem:

Given: Position of base plate and leg lengths.

Wanted: Position of the end plate.

Contrary to the inverse kinematics of the robot arm, the inverse problem of a parallel robot is trivial: given the end and base plate, we can immediately connect the legs. It is the forward displacement problem which admits multiple nontrivial solutions. We denote the coordinates of the base plate by $\mathbf{a}_i$, those of the end plate by $\mathbf{b}_i$, and the lengths of the legs by L_i, for $i = 1, 2, \ldots, 6$. At any particular position $\mathbf{p}$ and rotation R of the end plate, there will be unique squared distances:

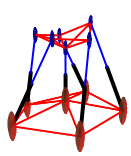
$$L_i^2 = (\mathbf{p} + R\mathbf{b}_i - \mathbf{a}_i)^T(\mathbf{p} + R\mathbf{b}_i - \mathbf{a}_i), \quad i = 1, 2, \ldots, 6. \tag{4}$$

To formulate the equations for we follow [4], using soma coordinates $[\mathbf{e} : \mathbf{g}] = [e_0 : e_1 : e_2 : e_3 : g_0 : g_1 : g_2 : g_3] \in \mathbb{P}^7$ quaternions on the Study quadric, the first equation in the system below:

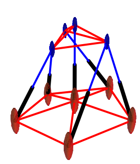
$$f(\mathbf{e}, \mathbf{g}) = \begin{cases} e_0 g_0 + e_1 g_2 + e_2 g_2 + e_3 g_3 = 0 \\ \mathbf{g} * \mathbf{g}' + (\mathbf{b}_i * \mathbf{b}_i' + \mathbf{a}_i * \mathbf{a}_i' - L_i^2)\mathbf{e} * \mathbf{e}' + (\mathbf{g} * \mathbf{b}_i' * \mathbf{e}' + \mathbf{e} * \mathbf{b}_i * \mathbf{g}') \\ \quad - (\mathbf{g} * \mathbf{e}' * \mathbf{a}_i' + \mathbf{a}_i * \mathbf{e} * \mathbf{g}') - (\mathbf{e} * \mathbf{b}_i * \mathbf{e}' * \mathbf{a}_i' + \mathbf{a}_i * \mathbf{e} * \mathbf{b}_i' * \mathbf{e}') = 0, i = 1, 2, \ldots 6 \end{cases} \tag{5}$$

where $e' = (e_0, -e_1, -e_2, -e_3)$ and $*$ is the multiplication of quaternions. Since the equations in $f(\mathbf{e}, \mathbf{g}) = \mathbf{0}$ are homogeneous, we augment the system with one linear equation with random complex coefficients.

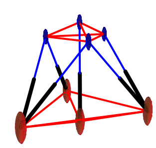
While for 7 quadratic equations we might expect 128 solutions, there are only 40 solutions to this system [4]. As the coordinates of the base and end plate are specialized to form coincident joints (as illustrated in the figure below), the number of solutions can only go down.



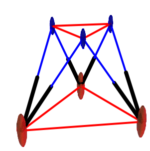
6-6, 40 solutions



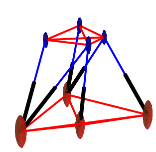
4-6, 32 solutions



4-4a, 16 solutions



3-3, 16 solutions



4-4b, 24 solutions

The coefficient-parameter homotopy has 42 parameters:

$$f(\mathbf{e}, \mathbf{g}, (1-t)\boldsymbol{\mu}_0 + t\boldsymbol{\mu}_1) = \mathbf{0}, \tag{6}$$

where $\boldsymbol{\mu}_0, \boldsymbol{\mu}_1 \in \mathbb{C}^{42}$.

3 Parameter Continuation

Consider a cubic polynomial f in x, with two parameters λ_1 and λ_2:

$$f(x, \lambda_1, \lambda_2) = x^3 + \left(-2\lambda_1 - \frac{11}{4}\lambda_2\right) x^2 + \left(\frac{11}{3}\lambda_1 + \frac{39}{8}\lambda_2\right) x - 2\lambda_1 - \frac{45}{16}\lambda_2. \tag{7}$$

The space of monic cubic polynomials $x^3 + c_2x^2 + c_1x + c_0$ is of dimension three: $(c_0, c_1, c_2) \in \mathbb{C}^3$. The parameter space for f is a plane with parameter representation

$$c_0 = -2\lambda_1 - \frac{45}{16}\lambda_2, \quad c_1 = \left(\frac{11}{3}\lambda_1 + \frac{39}{8}\lambda_2\right), \quad c_2 = \left(-2\lambda_1 - \frac{11}{4}\lambda_2\right). \tag{8}$$

We consider a path from $(\lambda_1, \lambda_2) = (3, 0)$ to $(\lambda_1, \lambda_2) = (0, 2)$:

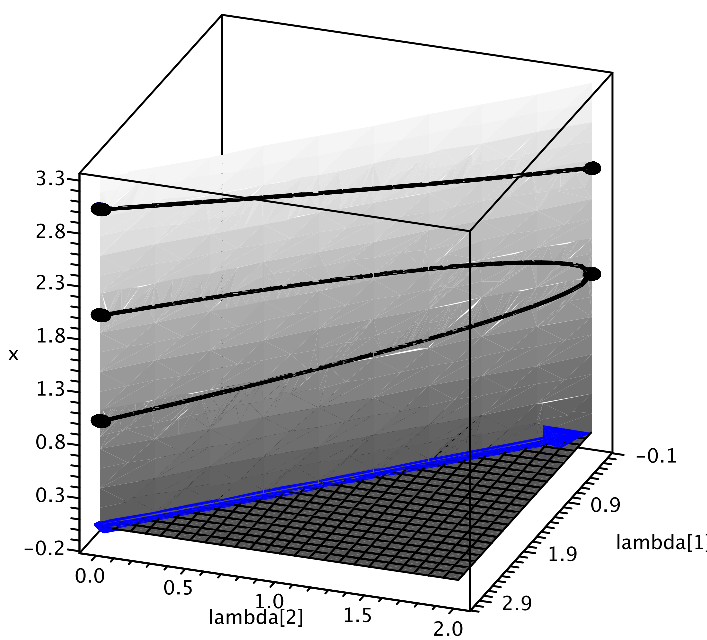
$$f(x, 3, 0) = x^3 - 6x^2 + 11x - 6 = (x-3)(x-2)(x-1) \tag{9}$$

$$f(x, 0, 2) = x^3 - \frac{11}{2}x^2 + \frac{39}{4}x - \frac{45}{8} = \left(x - \frac{3}{2}\right)^2 \left(x - \frac{5}{2}\right) \tag{10}$$

executed via the homotopy

$$f\left(x, \begin{bmatrix} 3 \\ 0 \end{bmatrix}(1-t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} t\right) = x^3 + \left(-6 + \frac{1}{2}t\right)x^2 + \left(11 - \frac{5}{4}t\right)x - 6 + \frac{3}{8}t. \tag{11}$$

At $t = 0$ we have 3 distinct roots of (9) and at $t = 1$ we have one double and one single root of (10). The plot below shows a path through the horizontal parameter space (λ_1, λ_2) with the incident solution space x shown vertically. At $t = 0$ we start with 3 distinct roots for $(\lambda_1, \lambda_2) = (3, 0)$. At $t = 1$ we end with one double and one single root for $(\lambda_1, \lambda_2) = (0, 2)$.



From the several versions of the theorems on parameter continuation in [3], we state the following theorem.

Theorem 3.1 (Parameter Continuation) *Consider $f(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{0}$ a polynomial system with side conditions on $\mathbf{x}$: $\mathbf{x} \in U \subset \mathbb{C}^n$ and parameter space $\boldsymbol{\lambda} \in Q \subset \mathbb{C}^n$. Then:*

1. *for generic $\boldsymbol{\lambda} \in Q$: $\mathcal{N}(\lambda, U)$ is the same finite constant;*
2. *for all $\boldsymbol{\lambda}^* \in Q$; $\mathcal{N}(\boldsymbol{\lambda}^*, U) \leq \mathcal{N}(\boldsymbol{\lambda}, U)$, with generic $\boldsymbol{\lambda} \in Q$,*
   moreover the $\boldsymbol{\lambda}^*$ for which $\mathcal{N}(\boldsymbol{\lambda}^*, U) < \mathcal{N}(\boldsymbol{\lambda}, U)$ belong to an algebraic subset $Q^* \subset Q$, $Q^* \neq Q$;
3. *all solutions paths starting at regular solutions in the homotopy $h(\mathbf{x}, t) = f(\mathbf{x}, (1-t)\boldsymbol{\lambda}_0 + t\boldsymbol{\lambda}_1) = \mathbf{0}$, for $t \in [0, 1]$, remain regular for generic choices of $\boldsymbol{\lambda}_0$ and $\boldsymbol{\lambda}_1$.*

For a sequence of nested parameter spaces $Q \supset Q_1 \supset Q_2 \supset \cdots$ repeated application of parameter continuation leads to a cascade of homotopies. After each stage in the cascade, the number of solutions is likely to decline and we need to filter out the spurious solutions before we start the next stage.

The *Zariski closure* $\overline{S}$ of a set S is the smallest algebraic set that contains S: $\overline{S} = V(I(S))$.

Elimination is a natural example of Zariski closure.

Theorem 3.2 *Consider $V = V(f_1, f_2, \ldots, f_N) \subset \mathbb{C}^n$ and let $\pi_\ell : \mathbb{C}^n \to \mathbb{C}^{n-\ell}$ be the projection onto the last $n - \ell$ components. If $I_\ell = \langle f_1, f_2, \ldots, f_N \rangle \cap \mathbb{C}[x_{\ell+1}, \ldots, x_n]$, then $V(I_\ell) = \overline{\pi_\ell(V)}$.*

The proof of this theorem (see [1]) needs the Hilbert Nullstellensatz.

4 using phc

Coefficient-parameter continuation is available via `phc -p`. If the user gives `phc -p` a system with more variables than unknowns, the user is prompted to choose between a complex homotopy or a sweep (with singularity detection).

We illustrate the use of `phc -p` on the illustrative example of [3, § 7.3], illustrating the law of cosines for an triangle. The input file to `phc -p` should contain the following:

```
2 5
 cos**2 + sin**2 - 1;
 (a*cos - b)*(a*cos - b) + a**2*sin**2 - c**2;

THE SOLUTIONS :
1 5
===========================================================
solution 1 :
t :  0.00000000000000E+00   0.00000000000000E+00
m : 1
the solution for t :
 cos : 0.8 0.0
 sin : 0.6 0.0
 a : 5.0 0.0
 b : 4.0 0.0
 c : 3.0 0.0
== err :  7.348E-16 = rco :  1.333E-01 = res :  0.000E+00 ==
===========================================================
```

The solution was obtained for specific values of the parameters `a`, `b`, and `c`, applying the blackbox solver.

The variables are the sine and the cosine of the angle between sides with lengths `a` and `b`.

5 Sard's Theorem and Cheater's Homotopy

A variant on the coefficient-parameter homotopies is known as the cheater's homotopy [2].

Theorem 5.1 (Sard's theorem) *Let $f(\mathbf{x}) = \mathbf{0}$ be a system of n equation in n unknowns. For a generic choice of $\mathbf{c} \in \mathbb{C}^n$, all solutions to the system $f(\mathbf{x}) + \mathbf{c} = \mathbf{0}$ are regular.*

The cheater's homotopy then in general is

$$h(\mathbf{x}, t) = f(\mathbf{x}, (1-t)\boldsymbol{\lambda}_0 + t\boldsymbol{\lambda}_1) + \mathbf{c} = \mathbf{0}, \quad \mathbf{c} \in \mathbb{C}^n, \quad t \in [0,1]. \tag{12}$$

The disadvantage of this homotopy might be that the number of solutions increases. The advantage is that this homotopy enables to compute the solutions with zero components, using the homotopy

$$h(\mathbf{x}, t) = f(\mathbf{x}, \boldsymbol{\lambda}_1) + (1-t)\mathbf{c} = \mathbf{0}, \quad \mathbf{c} \in \mathbb{C}^n, \quad t \in [0,1], \tag{13}$$

to remove the extra random constants $\mathbf{c} \in \mathbb{C}^n$. In some applications, solutions with zero components are irrelevant and the ability to exclude them is then beneficial. But for other applications, if one needs to know also the solutions with zero components, the cheater's homotopy is useful.

The cheater's homotopy reflects the cheating in the claim that systems with generic values of the parameters are easier to solve. With bootstrapping, we can use the result of other homotopies to start a coefficient-parameter or cheater's homotopy. However, even though linear-product start system may reduce the number of solution paths to be tracked, the number $\mathcal{N}(\lambda, U)$ is often much lower than any bound we can compute formally.

We will later see how polyhedral homotopies will remove the cheating when the parameter space coincides with the coefficient space of the polynomial systems.

6 Exercises

1. For the parameter homotopy in (11), compute the values for t for which the corresponding solutions are multiple.

2. Modify the setup for (11) to illustrate that when restricted to real paths turning points start to appear.

3. Maple code for the multiplication $*$ of quaternions is below

```
CreateQuaternion := (s,v) -> s + v[1]*i + v[2]*j + v[3]*k:
RealPart := q -> coeff(coeff(coeff(q,i,0),j,0),k,0):
VectorPart := q -> <coeff(q,i,1),coeff(q,j,1),coeff(q,k,1)>:
Conjugate := q -> RealPart(q) - coeff(q,i,1)*i - coeff(q,j,1)*j - coeff(q,k,1)*k:
MultiplyQuaternions := proc(q,r)
  description ‘multiplies two quaternions‘:
  local q0,r0,u,v,s,w:
  q0 := RealPart(q): u := VectorPart(q):
  r0 := RealPart(r): v := VectorPart(r):
  s := r0*q0 - u[1]*v[1] - u[2]*v[2] - u[3]*v[3]:
  w := <q0*v[1]+r0*u[1],q0*v[2]+r0*u[2],q0*v[3]+r0*u[3]>:
  w[1] := w[1] + (u[2]*v[3] - v[2]*u[3]):
  w[2] := w[2] - (u[1]*v[3] - v[1]*u[3]):
  w[3] := w[3] + (u[1]*v[2] - v[1]*u[2]):
  return CreateQuaternion(s,w):
end proc:
```

Use Maple or Sage, or any other computer algebra system to define the equations of (5).

4. Solve the system (5) defined above using `phc`.

5. Use the solutions of the system (5) as start solutions in a coefficient-parameter homotopy to solve the 3-3 configuration of the platform: the joints on both base and end platforms are coincided into triangles.

References

[1] D. Cox, J. Little, and D. O'Shea. *Ideals, Varieties and Algorithms. An Introduction to Computational Algebraic Geometry and Commutative Algebra*. Undergraduate Texts in Mathematics. Springer–Verlag, second edition, 1997.

[2] T.Y. Li, T. Sauer, and J.A. Yorke. The cheater's homotopy: an efficient procedure for solving systems of polynomial equations. *SIAM J. Numer. Anal.*, 26(5):1241–1251, 1989.

[3] A.J. Sommese and C.W. Wampler. *The Numerical solution of systems of polynomials arising in engineering and science*. World Scientific Press, Singapore, 2005.

[4] C.W. Wampler. Forward displacement analysis of general six-in-parallel SPS (Stewart) platform manipulators using soma coordinates. *Mech. Mach. Theory*, 31(3):331–337, 1996.