

## Kushnirenko's Theorem

Most applications arising in practical applications are sparse in the sense that not all monomials (up to the degree of the polynomial) appear with a nonzero coefficient. The sparser a system, the fewer isolated solutions we expect. We consider the design of 4-bar mechanisms. After defining homotopies associated to regular triangulations, we end by stating Kushnirenko's theorem.

### 1 Binomial Systems

The sparsest polynomial systems that we can solve in a straightforward manner have exactly two monomials in every equation. These systems are called binomial systems.

Denote  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ . A matrix  $A$  with integer coefficients and a vector  $\mathbf{c} \in (\mathbb{C}^*)^n$  define a *binomial system*, denoted as  $\mathbf{x}^A = \mathbf{c}$ . The columns of  $A$  define the exponent vectors of the equations in the binomial system, i.e.: for  $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n]$ , we have

$$\mathbf{x}^A = \mathbf{x}^{[\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n]} = [\mathbf{x}^{\mathbf{a}_1} \ \mathbf{x}^{\mathbf{a}_2} \ \cdots \ \mathbf{x}^{\mathbf{a}_n}] \quad \text{and} \quad \mathbf{x}^A = \mathbf{c} \Leftrightarrow [\mathbf{x}^{\mathbf{a}_1} = c_1 \ \mathbf{x}^{\mathbf{a}_2} = c_2 \ \cdots \ \mathbf{x}^{\mathbf{a}_n} = c_n]. \quad (1)$$

Consider for example the following binomial system:

$$\begin{cases} x_1^3 x_2^2 = 1 \\ x_1^5 x_2^1 = 1 \end{cases} \quad \Leftrightarrow \quad \mathbf{x}^A = \mathbf{c} : [x_1 \ x_2] \begin{bmatrix} 3 & 5 \\ 2 & 1 \end{bmatrix} = [1 \ 1]. \quad (2)$$

To make  $A$  upper triangular, we define a unimodular matrix  $M$  taking the greatest common divisor of 3 and 2:  $\gcd(3, 2) = 1 = (+1) \cdot 3 + (-1) \cdot 2$ . To eliminate the 2 in  $A$ , the second row in  $M$  will contain the coefficients of the linear combination of 3 and 2:  $(+2) \cdot 3 + (-3) \cdot 2 = 0$ . So we have

$$M = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}, \det(M) = 1 \quad MA = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & -7 \end{bmatrix} = U. \quad (3)$$

The unimodular transformation  $M$  defines a change of coordinates:

$$\mathbf{y}^M = \mathbf{x} \Rightarrow \mathbf{x}^A = \mathbf{y}^{MA} = \mathbf{y}^U. \quad (4)$$

More explicitly, the coordinate transformation is

$$[x_1 \ x_2] = [y_1 \ y_2] \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = [x_1 = y_1 y_2^{-2} \quad x_2 = y_1^{-1} y_2^3]. \quad (5)$$

When doing the coordinate transformation on the system, we recognize the matrix multiplication  $MA$ :

$$\begin{cases} x_1^3 x_2^2 = (y_1 y_2^{-2})^3 (y_1^{-1} y_2^3)^2 = y_1^{1 \cdot 3 + (-1) \cdot 2} y_2^{(-2) \cdot 3 + 3 \cdot 2} = y_1 \\ x_1^5 x_2^1 = (y_1 y_2^{-2})^5 (y_1^{-1} y_2^3)^1 = y_1^{1 \cdot 5 + (-1) \cdot 1} y_2^{(-2) \cdot 5 + 3 \cdot 1} = y_1^4 y_2^{-7} \end{cases} \quad (6)$$

The change of coordinates has not altered the number of solutions, which equals

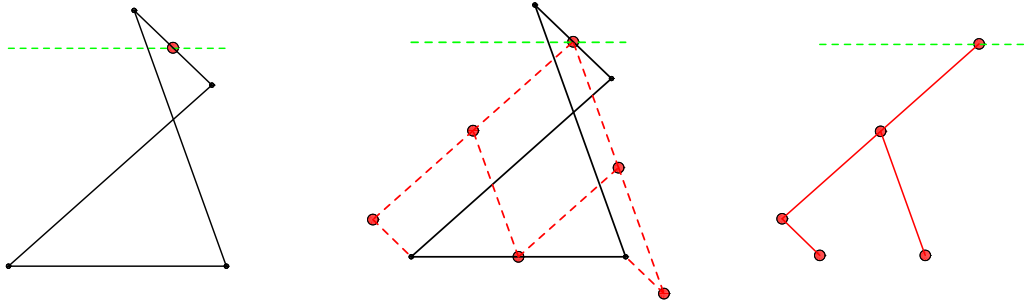
$$|\det(U)| = |\det(MA)| = |\det(U) \det(A)| = |\det(A)| = 7. \quad (7)$$

The upper triangular matrix  $U$  is the Hermite normal form of the matrix  $A$ . The unimodular transformations  $M$  are the discrete analogues to Givens rotations. Making the exponent matrix upper triangular defines coordinate transformations to make the binomial system triangular.

**Theorem 1.1 (regularity of a binomial system)** *If  $\det(A) \neq 0$  and  $\mathbf{c} \in (\mathbb{C}^*)^n$ , then  $\mathbf{x}^A = \mathbf{c}$  has exactly as many as  $|\det(A)|$  isolated regular solutions in  $(\mathbb{C}^*)^n$ .*

## 2 Design of 4-bar Mechanisms

A 4-bar mechanism consists of 4 rigid bars, attached to each other by joints. A special 4-bar mechanism, attributed to Chebyshev, see [4], is displayed below:



At the left we see the original configuration of four bars. The big dot at the top bar traces the dashed line. The picture in the middle shows that every 4-bar mechanism occurs in a group of three. At the right is one of these cognates, as it is often used in practice. The mechanism translates horizontal motion at the top into circular motion at the bottom left joint.

The design of a 4-bar mechanism can be formulated as an interpolation problem. Given the coordinates of the points on the coupler curve, find the lengths of the bars of the linkage so that the coupler curve passes through the given points.

It is convenient to use complex representations of the points: A point  $(a, b) \in \mathbb{R}^2$  is mapped to  $z = a + ib$ ,  $i = \sqrt{-1}$ . Then  $(z, \bar{z}) = (a + ib, a - ib) \in \mathbb{C}^2$  are *isotropic coordinates*. Observe  $z \cdot \bar{z} = a^2 + b^2$ . Rotation around  $(0, 0)$  through angle  $\theta$  is multiplication by  $e^{i\theta}$ . Multiply by  $e^{-i\theta}$  to invert the rotation. If we abbreviate a rotation by  $\Theta = e^{i\theta}$ , then its inverse  $\Theta^{-1} = \bar{\Theta}$ , satisfying  $\Theta\bar{\Theta} = 1$ .

The loop equations [3] are formulated as follows. Let  $A = (a, \bar{a})$  and  $B = (b, \bar{b})$  be the fixed base points. Unknown are  $(x, \bar{x})$  and  $(y, \bar{y})$ , coordinates of the other two points in the 4-bar linkage. For given precision points  $(p_j, \bar{p}_j)$ , assuming  $\theta_0 = 1$ ,

$$\begin{cases} (p_j + x\theta_j + a)(\bar{p}_j + \bar{x}\bar{\theta}_j + \bar{a}) = (p_0 + x + a)(\bar{p}_0 + \bar{x} + \bar{a}) \\ (p_j + y\theta_j + b)(\bar{p}_j + \bar{y}\bar{\theta}_j + \bar{b}) = (p_0 + y + b)(\bar{p}_0 + \bar{y} + \bar{b}) \end{cases} \quad j = 1, 2, 3, 4. \quad (8)$$

Since the angle  $\theta_j$  corresponding to each  $(p_j, \bar{p}_j)$  is unknown, five precision points are needed to determine the linkage uniquely. Adding  $\theta_j\bar{\theta}_j = 1$  to the system leads to 12 equations in 12 unknowns:  $(x, \bar{x})$ ,  $(y, \bar{y})$ , and  $(\theta_j, \bar{\theta}_j)$ , for  $j = 1, 2, 3, 4$ .

Using  $\theta_j\bar{\theta}_j = 1$ , we may eliminate  $\bar{\theta}_j$  via  $\bar{\theta}_j = \theta_j^{-1}$ ,  $j = 1, 2, 3, 4$ . Then we see the eight equations in (8) as four pairs of linear equations in  $\theta_j$  and  $\theta_j^{-1}$ . Applying Cramer's rule (as suggested in [3]) to

$$\begin{cases} \alpha_1\theta + \alpha_2\theta^{-1} + \alpha_3 = 0 \\ \beta_1\theta + \beta_2\theta^{-1} + \beta_3 = 0 \end{cases} \quad (9)$$

we may eliminate  $\theta$  and  $\theta^{-1}$  from the system. The system (9) has solutions only if  $\delta_1\delta_2 + \delta_3^2 = 0$  where

$$\delta_1 = \det \begin{pmatrix} \alpha_2 & \alpha_3 \\ \beta_2 & \beta_3 \end{pmatrix}, \quad \delta_2 = \det \begin{pmatrix} \alpha_1 & \alpha_3 \\ \beta_1 & \beta_3 \end{pmatrix}, \quad \text{and} \quad \delta_3 = \det \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix}. \quad (10)$$

With this elimination, the system of 12 equations in 12 unknowns reduces to a system of 4 equations in 4 unknowns. All equations in the system share the same support.

### 3 Regular Triangulations and Homotopies

The support  $A$  of a polynomial  $f$  is a finite set of exponent vectors which models the sparse structure of  $f$ . In particular, we write

$$f(\mathbf{x}) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad c_{\mathbf{a}} \in \mathbb{C}^*, \quad \mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}. \quad (11)$$

The convex hull of  $A$  is the Newton polytope of  $f$ , denoted by  $Q = \text{conv}(A)$ . A really cool book to learn more about polytopes is [6], although we will mainly stick to polygons, i.e.:  $n = 2$ .

In this lecture, we consider systems  $f(\mathbf{x}) = \mathbf{0}$  whose equations all share the same support  $A$ , or a bit more generally, the same Newton polytope  $Q$ . For generic choices of the coefficients, the monomials whose exponent vector is not a vertex do not have an influence on the volume of  $Q$  and may be omitted.

To compute the volume of a polytope we need a triangulation  $\Delta$ :

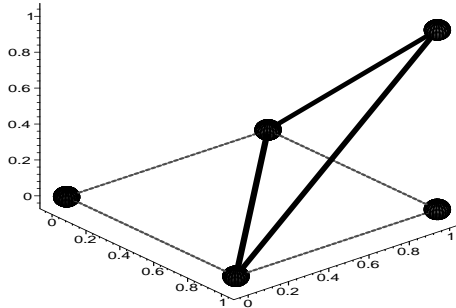
$$\text{vol}(Q) = \sum_{S \in \Delta} \text{vol}(S) \quad \text{or} \quad \text{vol}(Q) = \sum_{C \in \Delta} \text{vol}(C), \quad S = \text{conv}(C), \quad \#C = n + 1. \quad (12)$$

The cells  $C$  in a triangulation span simplices  $S$ . We normalize our volumes so that the unit simplex has volume 1, instead of  $1/n!$ . Triangulations that are of interest to us are induced by the application of a lifting function  $\omega$ :

$$\omega : \mathbb{Z}^n \rightarrow \mathbb{Z} : \mathbf{a} \mapsto \omega(\mathbf{a}), \quad (13)$$

which embeds the support  $A$  into  $\mathbb{Z}^{n+1}$ ,  $\hat{A} = \omega(A)$ . Accordingly:  $\hat{Q} = \text{conv}(\hat{A})$ . A triangulation is *regular* if there is a lifting function so there is a 1-to-1 mapping of the facets of the lower hull of the lifted point configuration and the simplices in the triangulation.

The picture below shows a regular triangulation of the unit square:



This unit square is the Newton polytope of the system  $\hat{g}(\mathbf{x}, t) = \mathbf{0}$ :

$$\hat{g}(\mathbf{x}, t) = \begin{cases} c_{1,11}x_1x_2t^1 + c_{1,10}x_1t^0 + c_{1,01}x_2t^0 + c_{1,00}t^0 = 0 \\ c_{2,11}x_1x_2t^1 + c_{2,10}x_1t^0 + c_{2,01}x_2t^0 + c_{2,00}t^0 = 0. \end{cases} \quad (14)$$

The powers of  $t$  are given by the lifting function  $\omega$ . We see immediately how at  $\hat{g}(\mathbf{x}, t = 0) = \mathbf{0}$  we find one start solution, as solution of the system supported by the cell  $\{(0, 0), (1, 0), (0, 1)\}$ .

To construct the homotopy that starts at system supported at the other cell of the triangulation, we look at the inner normal of the lifted cell, i.e.:  $\mathbf{v} = (-1, -1, +1)$ . This inner normal defines the change of coordinates  $x_1 = y_1 t^{-1}$  and  $x_2 = y_2 t^{-1}$ . After multiplication by  $t$ , this change of coordinates yields

$$\hat{g}(\mathbf{y}, t) = \begin{cases} c_{1,11}y_1y_2t^0 + c_{1,10}y_1t^0 + c_{1,01}y_2t^0 + c_{1,00}t^1 = 0 \\ c_{2,11}y_1y_2t^0 + c_{2,10}y_1t^0 + c_{2,01}y_2t^0 + c_{2,00}t^1 = 0. \end{cases} \quad (15)$$

At  $\hat{g}(\mathbf{y}, t = 0) = \mathbf{0}$  there is another solution which contributes one path.

## 4 Puiseux Series and Kushnirenko's Theorem

The development of the solutions of a polyhedral homotopy in terms of series with fractional powers relies on the theorem of Puiseux [5]. By  $\mathbb{C}(t)^*$ , we denote the field of fractional power series with complex coefficients. An element in  $\mathbb{C}(t)^*$  is a series in  $t$  which may have negative and fractional exponents. The short form of the theorem of Puiseux as in [5] says that the field  $\mathbb{C}(t)^*$  is algebraically closed. Fractional power series are also called Puiseux series. We give the longer form of the theorem of Puiseux below.

**Theorem 4.1 (the theorem of Puiseux)** *Let  $f(x_1, x_2) \in \mathbb{C}(x_2)[x_1]$ :  $f$  is a polynomial in the variable  $x_1$  and its coefficients are fractional power series in  $x_2$ . The polynomial  $f$  has as many series solutions as the degree of  $f$ . Every series solution has the following form:*

$$\begin{cases} x_1 = t^u \\ x_2 = ct^v(1 + O(t)), \quad c \in \mathbb{C}^* \end{cases} \quad (16)$$

where  $(u, v)$  is an inner normal to an edge of the lower hull of the Newton polygon of  $f$ .

The algorithm to compute all Puiseux series solutions is very similar to a polyhedral homotopy. One can view the theorem of Puiseux as a generalization of the fundamental theorem of algebra. The computation of the inner normals followed by the application of Newton-Hensel lifting to compute the Puiseux series is called the Newton-Puiseux method.

At first, the theorem of Kushnirenko applies to a more restricted class of polynomial systems, because all polynomials in the system are expected to have the same support. Despite this, this theorem is often already much more useful in practice than Bézout's theorem.

**Theorem 4.2 (Kushnirenko [2])** *Consider the system  $f(\mathbf{x}) = \mathbf{0}$  and let  $A$  be the support of every polynomial in  $f$ . Then the number of isolated solutions of  $f(\mathbf{x}) = \mathbf{0}$  in  $(\mathbb{C}^*)^n$  cannot exceed the volume of the polytope spanned by  $A$ .*

Following [1], polyhedral homotopies provide a constructive proof of Kushnirenko's theorem. The constructive proof uses the following two arguments:

1. For sufficiently generic choices of the coefficients, polyhedral homotopies track as many paths as the volume of the Newton polytope of the system, as  $t$  goes from 0 to 1. Any regular triangulation defines polyhedral homotopies with exactly as many paths as the volume of the Newton polytope.
2. Considering a polyhedral homotopy as  $t$  goes from 1 to 0, there can be no more solutions than the volume of the Newton polytope. For this argument we need a more refined concept of infinity.

## 5 Polyhedral Algorithms

To solve binomial systems of the form  $\mathbf{x}^A = \mathbf{c}$  we need the Hermite normal form of the matrix  $A$ . We make zeroes in a matrix via unimodular transformations:

$$\begin{bmatrix} k & \ell \\ -\frac{b}{d} & \frac{a}{d} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} d \\ 0 \end{bmatrix} \quad (17)$$

where

$$\gcd(a, b) = ka + \ell b = d. \quad (18)$$

Because the determinant of a product is the product of determinants, the product of unimodular matrices is again unimodular. Let  $M_{ij}$  be the unimodular matrix to make the  $(i, j)$ -th element of a matrix zero, then (without pivoting):

$$M = M_{nn-1} \cdots M_{n2} \cdots M_{32} M_{n1} \cdots M_{31} M_{21}, \quad MA = U \tag{19}$$

where  $U$  is upper triangular, the Hermite normal form of  $A$ .

As we apply the Hermite normal form to solve  $\mathbf{x}^A = \mathbf{c}$ , the columns of  $A$  are obtained as cells that span the simplices in a triangulation. Next we outline a method to compute a regular triangulation, placing the points one after the other and updating the cells in the triangulation via pivoting. One key observation is that if we add a new point to a triangulation, we can always generate a lifting of that new point high enough so the cells already computed in the existing triangulation stay as lower facets of the lower hull.

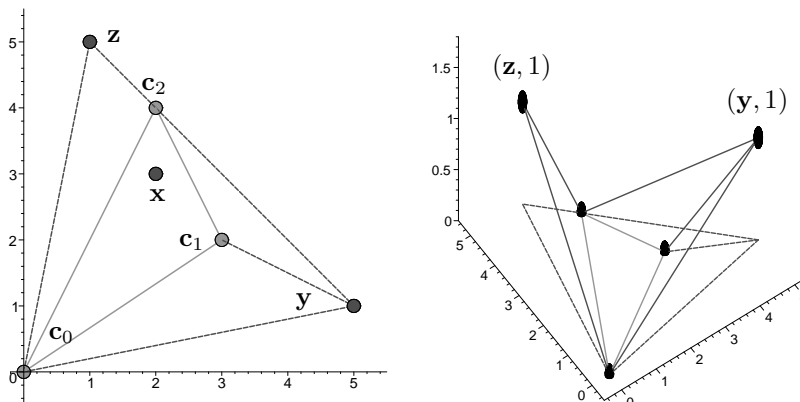
An algorithm to compute a regular triangulation, placing points is sketched below:

0.  $n + 1$  linearly independent points span the first cell.
1. For all remaining points:
  - (a) check if the point is inside the interior of a cell;
  - (b) if outside, then apply pivoting to find new cells.

Let  $[\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2]$ , with  $\mathbf{c}_0 = (0, 0)$ ,  $\mathbf{c}_1 = (3, 2)$ , and  $\mathbf{c}_2 = (2, 4)$ .

point	barycentric decomposition	pivoting
$\mathbf{x} = (2, 3)$ :	$\mathbf{x} = +\frac{1}{8}\mathbf{c}_0 + \frac{1}{4}\mathbf{c}_1 + \frac{5}{8}\mathbf{c}_2$	no new simplex
$\mathbf{y} = (5, 1)$ :	$\mathbf{y} = -\frac{1}{3}\mathbf{c}_0 + \frac{9}{4}\mathbf{c}_1 - \frac{7}{8}\mathbf{c}_2$	$[\mathbf{y}, \mathbf{c}_1, \mathbf{c}_2][\mathbf{c}_0, \mathbf{c}_1, \mathbf{y}]$
$\mathbf{z} = (1, 5)$ :	$\mathbf{z} = +\frac{1}{8}\mathbf{c}_0 - \frac{3}{4}\mathbf{c}_1 + \frac{13}{8}\mathbf{c}_2$	$[\mathbf{c}_0, \mathbf{z}, \mathbf{c}_2]$

The picture below shows the cases that may occur when pivoting.



The construction on the right shows how the triangulation can be obtained as the lower hull of  $\mathbf{y}$  and  $\mathbf{z}$  lifted at height one, with  $[\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2]$  sitting at level zero. While our geometric intuition is limited to 3-space, using linear algebra, the method to place points into a regular triangulation works for any dimension.

The complexity of volume computation is #P-hard because there is no polynomial time algorithm to verify whether some given number is the correct volume. As we took the volume of the unit simplex as one, the unit cube has volume  $n!$  relating the Kushnirenko bound to the permanent of an integer matrix.

## 6 Exercises

1. Consider the binomial system  $\mathbf{x}^A = \mathbf{c}$  with

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = [1 \ 1 \ 1]. \quad (20)$$

Solve this system. How many solutions does this system have?

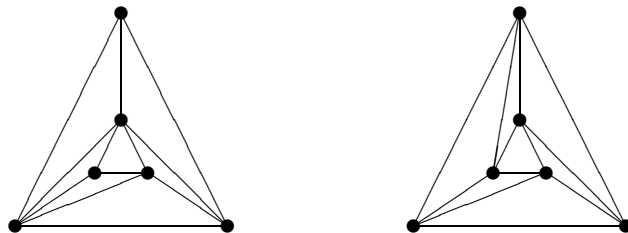
Compare this number with the lowest Bézout bound.

2. Make a Maple worksheet or Sage notebook to generate the system defined by (8).

Perform the elimination of the  $\theta$  variables using Cramer's rule.

Solve the polynomial system and verify that for random choices of the parameters there are indeed as many solutions as the volume of the Newton polytope.

3. Consider  $A = \{(0, 0), (8, 0), (4, 8), (3, 2), (4, 4), (5, 2)\}$  and two triangulations below:



Show that the triangulation on the left is regular, while the one on the right is not.

4. Instead of defining the support of a polynomial as the collection of exponents for which the corresponding coefficient is different from zero, we can first prescribe the support  $A$  and then consider all polynomials whose support is a subset of  $A$ .

(a) What is the prescribed support and Newton polytope of the theorem of Bézout? Give an example in 2 variables and interpret Bézout's theorem by the application of Kushnirenko's theorem.

(b) Do the same for the multihomogeneous version of Bézout.

5. Give an example of a class of systems of two polynomial equations in two variables with shared support so that the area of the Newton polygon is much less than the product of the degrees or the 2-homogeneous Bézout bound. What seems to be typical for such systems?

6. Consider the homotopy

$$h(x_1, x_2, t) = \begin{cases} x_1 x_2 + x_1 - x_2 + t = 0 \\ x_1 x_2 - 2x_1 + x_2 + t = 0 \end{cases} \quad (21)$$

(a) The series solution up to fourth order is

$$\left( \frac{1}{3} - 2t - 12t^2 - 144t^3 + O(t^4), \frac{1}{2} - 3t - 18t^2 - 216t^3 + O(t^4) \right). \quad (22)$$

Develop the series solution further up to  $O(t^8)$ .

(b) Compute the other series solution for this system, using a proper coordinate transformation.

## References

- [1] B. Huber and B. Sturmfels. A polyhedral method for solving sparse polynomial systems. *Math. Comp.*, 64(212):1541–1555, 1995.
- [2] A.G. Kushnirenko. Newton polytopes and the Bézout theorem. *Functional Anal. Appl.*, 10(3):233–235, 1976. Translated from *Funktsional. Anal. i Prilozhen.* 10(3),82–83,1976.
- [3] A.J. Sommese and C.W. Wampler. *The Numerical solution of systems of polynomials arising in engineering and science*. World Scientific Press, Singapore, 2005.
- [4] K.J. Waldron and G.L. Kinzel. *Kinematics, Dynamics, and Design of Machinery*. John Wiley & Sons, second edition, 2003.
- [5] R.J. Walker. *Algebraic Curves*. Princeton University Press, 1950.
- [6] G.M. Ziegler. *Lectures on Polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.