Standard Bases

In this lecture we look at singular solutions, following [3] and [5]. We define the multiplicity of an isolated solution geometrically and algebraically. To compute the multiplicity of singular points, we work in local rings and construct standard bases.

1 Localization and Multiplicities

The localization of $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, x_2, \ldots, x_n]$ at $\langle \mathbf{x} \rangle = \langle x_1, x_2, \ldots, x_n \rangle$ is

$$\mathbb{C}[\mathbf{x}]_{\langle \mathbf{x} \rangle} = \left\{ \frac{f(\mathbf{x})}{g(\mathbf{x})} \mid f, g \in \mathbb{C}[\mathbf{x}], g(\mathbf{0}) \neq 0 \right\}. \quad (1)$$

Instead of at $\langle \mathbf{x} \rangle$ we can localize from any point $\mathbf{z} \in \mathbb{C}^n$, not just the origin $\mathbf{0}$, shifting the coordinate system $x_i = y_i - z_i$, $i = 1, 2, \ldots, n$. Using suitable monomial orderings (Singular: `ds`, `Ds`, and `ls`) we compute in $\mathbb{C}[\mathbf{x}]_{\langle \mathbf{x} \rangle}$ without denominators.

To motivate our interest in local properties, consider for example $I = \langle y(x-1), z(x-1) \rangle \subset \mathbb{Q}[x, y, z]$. The real picture of $V(I)$ is in Figure 1.

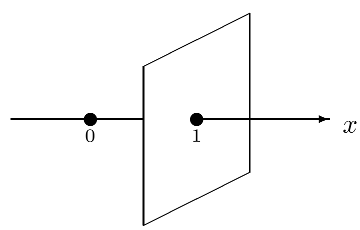

Figure 1: The real picture of $y(x-1) = 0$ and $z(x-1) = 0$. The local dimension of $V(I)$ at $(0, 0, 0)$ is 1 and is 2 at $(1, 0, 0)$.

In a local ring, $1 - x$ is a unit, i.e.: it has an inverse $\frac{1}{1-x} = 1 + x + x^2 + \cdots \in \mathbb{C}[[x]]$, as a formal power series. Working with local rings is similar to working with formal power series.

Consider the ideal $I = \langle x_1^2 + x_1^3, x_2^2 \rangle$. Factoring the first polynomial as $x_1^2(1 + x_1)$, we see that $V(I)$ consists of two distinct roots. To focus on the singular solution at the origin, we have to look at the lowest powers of the monomials instead of the highest ones. Therefore we order the terms in the opposite order, equivalent of taking negative weights.

For example, in lecture 4 we defined global term orders, among them $>_{\rm lex}$. The corresponding local term order neglex is $\mathbf{x}^\mathbf{a} >_{\rm neglex} \mathbf{x}^\mathbf{b}$ if the leftmost nonzero entry in $\mathbf{a} - \mathbf{b}$ is negative. For example: $x_1 x_2^3 >_{\rm neglex} x_1^2 x_2$.

With respect to a local term order, we may also derive a normal form algorithm and the analogue to a Gröbner basis. Such an analogue basis is called a standard basis. For the ideal I from above, a Gröbner basis leads to the monomial ideal $\langle x_1^3, x_2^2 \rangle$ which tells us there are six isolated solutions. A standard basis for I gives the monomial ideal $\langle x_1^2, x_2^2 \rangle$.

A Gröbner basis is global and when the solution set $V(I)$ consists of finitely many points, the dimension of the quotient ring $\mathbb{C}[\mathbf{x}]/I$ equals $\#V(I)$ and the monomials which do not belong to $\langle LT(I) \rangle$ define a basis for $\mathbb{C}[\mathbf{x}]/I$. Via a standard basis we define the multiplicity of zero as a point in $V(I)$ via the dimension of the quotient ring as computed with a standard basis analogously as with a Gröbner basis.

While the ideal I we took looks trivial, a less simple example is $I = \langle x^2, xy, y^2 \rangle$, at least numerically...

2 Singular Points on Curves and Surfaces

A point $\mathbf{z} \in \mathbb{C}^n$ on a hypersurface $f(\mathbf{x}) = 0$ is singular if in addition to $f(\mathbf{z}) = 0$ also all its partial derivatives vanish: $\frac{\partial f}{\partial x_i}(\mathbf{z}) = 0$, $i = 1, 2, \ldots, n$. The singularity is isolated if there exists a ball around the singular point that contains no other singular point.

The pictures below in Figure 2 are taken from [10].

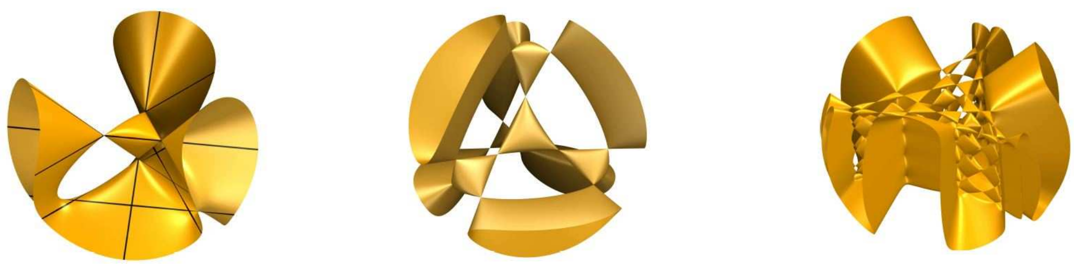

Figure 2: On the left is a picture of the four-nodal Cayley cubic and its nine lines. The picture on the middle is a 16-nodal Kummer surface, known to Kummer in 1864. This is a quartic surface in 3-space with the maximum number of nodes. On the right is a surface with 216 nodes, known as Breske's real variant of Chmutov's nonic. Chmutov's work discoveries date from 1992.

The affine equation for the four-nodal Cayley cubic is given in [10, page 26]:

$$4\left(x^3 + 3x^2 - 3xy^2 + 3y^2 + \frac{1}{2}\right) + 3\left(x^2 + y^2\right)(z - 6) - z\left(3 + 4z + 7z^2\right) = 0. \tag{2}$$

This surface contains exactly three lines of multiplicity one and six lines of multiplicity four.

Figure 3 shows a cuspidal cubic $f = x^3 - y^2 = 0$. If we intersect f with a general line we get three points in the complex plane, as f is defined by a cubic. The multiplicity of the cusp at $(0,0)$ equals two.

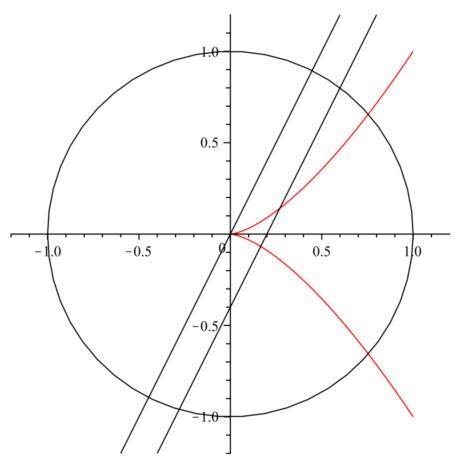

Figure 3: A cuspidal cubic given by $f = x^3 - y^2 = 0$ intersected by $x + \frac{1}{2}y = 0$ and $x + \frac{1}{2}y = \epsilon$. Restrict f to the shifted line $f(x = \epsilon - \frac{1}{2}y, y) = (\epsilon - \frac{1}{2}y)^3 - y^2 = -y^2(\frac{1}{8}y + 1) + \frac{3}{4}y^2\epsilon + O(\epsilon^2)$ and we find two zeroes close to $(0,0)$. Therefore we say the cusp has multiplicity two.

3 Mora's Normal Form Algorithm

Similar to Buchberger's algorithm to compute Gröbner bases, there is an algorithm to compute standard basis, originally by Mora and in more general form by Greuel and Pfister [5]. A more geometric name for Mora's normal form algorithm [11] is the tangent cone algorithm.

Instead of $x = \left(\sum_{k=0}^{\infty} x^k \right) (x - x^2)$ we write $(1-x)x = x - x^2$. This gives rise to a weak normal form:

$$uf = \sum_{i=1}^{s} a_i g_i + \mathrm{NF}(f|G), \quad G = \{g_1, g_2, \ldots g_s\}. \tag{3}$$

We define $\mathrm{ecart}(f) := \deg(f) - \deg(\mathrm{LM}(f))$. For a homogeneous polynomial f we have $\mathrm{ecart}(f) = 0$. Algorithm 3.1 returns a weak normal form of a polynomial with respect to a set of polynomials. This algorithm is then used in Algorithm 3.2 which is a reformulation of Buchberger's algorithm.

Algorithm 3.1 NFMora$(f, G, >)$

Input: $f \in \mathbb{C}[\mathbf{x}]$, $G = \{g_1, g_2, \ldots, g_s\}$, $g_i \in \mathbb{C}[\mathbf{x}], i = 1, 2, \ldots, s$;
    $>$ is any monomial ordering.
Output: $h \in \mathbb{C}[\mathbf{x}] : uf = \sum_{i=1}^{s} a_i g_i + h$.
$h := f; T := G$;
while $(h \neq 0)$ do
  $T_h := \{ \ g \in T : \mathrm{LM}(g) | \mathrm{LM}(h) \ \}$;
  if $T_h = \emptyset$ then
    return h;
  else
    choose $g \in T_h$ with minimal $\mathrm{ecart}(g)$;
    if $\mathrm{ecart}(g) > \mathrm{ecart}(h)$ then $T := T \cup \{h\}$;
    $h :=$ Spolynomial(h, g);
  end if;
end while.

Like with the division algorithm, we will need a basis for the outcome to be conclusive.

Algorithm 3.2 Standard$(G, \mathrm{NF}, >)$

Input: G is a finite list of polynomials,
    NF is an algorithm to compute a weak normal form
    $>$ is any monomial ordering.
Output: S is a standard basis for $\langle G \rangle$.
$S := G$; $P := \{ \ (f, g) \mid f, g \in S, f \neq g \ \}$;
while $P \neq \emptyset$ do
  $(f, g) :=$ pop from P;
  $h := \mathrm{NF}(\mathrm{Spolynomial}(f, g), S, >)$;
  if $h \neq 0$ then
    $P := P \cup \{ \ (h, f) \mid f \in S \ \}$;
    $S := S \cup \{h\}$;
  end if;
end while.

We copied the description of the algorithms from [5], correctness and termination is given in [11], see also [12], where they are also called Hironaka Bases. Standard bases originated in the resolution of singularities [8, 9] and are a tool in local analytic geometry [4], see [1], and [6, 7] for informal expositions.

Proposition 3.1 *Consider $I \subset \mathbb{C}[\mathbf{x}]$ and $\mathbf{0} \in \mathbb{C}^n$ is on an irreducible component W of $V(I)$. Let $G = \{g_1, g_2, \ldots, g_s\}$ be a standard basis of I with respect to some local order $>$. Consider $p \in \mathbb{C}[\mathbf{x}]$ and let $r = \mathrm{NFMora}(p, G, >)$. If $r = 0$, then $p \in I(W)$.*

Proof. If $r = 0$, then $up = a_1 g_1 + a_2 g_2 + \cdots + a_s g_s$, where $u \in \mathbb{C}[\mathbf{x}]$ is invertible in $\mathbb{C}[\mathbf{x}]_{\langle \mathbf{x} \rangle}$, i.e.: $u(0) \neq 0$. We have: $W \subset V(I) \Rightarrow up \in I(W)$. Because W is irreducible ($I(W)$ is prime):

$$up \in I(W) \Rightarrow u \in I(W) \text{ or } p \in I(W). \tag{4}$$

But since $u(0) \neq 0$, $u \neq I(W)$, thus $p \in I(W)$. □

A geometric application of standard bases is the computation of tangent cones, defined via Taylor series. Consider a point $\mathbf{p} \in \mathbb{C}^n$ and $f \in \mathbb{C}[x]$, $d = \deg(f)$. Developing f as Taylor series about $\mathbf{p}$, using the notations $|\mathbf{a}| = a_1 + a_2 + \cdots + a_n$ and $\mathbf{a}! = a_1! a_2! \cdots a_n!$:

$$\begin{aligned} f(x) &= f(\mathbf{p}) + \sum_{0<|\mathbf{a}|\leq d} \frac{1}{\mathbf{a}!} \frac{\partial^{\mathbf{a}} f}{\partial x_1^{a_1} \partial x_2^{a_2} \cdots \partial x_n^{a_n}}(\mathbf{p}) \prod_{i=1}^{n} (x_i - p_i)^{a_i} & (5) \\ &= f_{\mathbf{p},0} + f_{\mathbf{p},1} + \cdots + f_{\mathbf{p},d}. & (6) \end{aligned}$$

Denote by $f_{\mathbf{p},min}$ as in the Taylor series for f about the point $\mathbf{p}$: $f_{\mathbf{p},min}(\mathbf{p}) \neq 0$ and for all $j < min$: $f_{\mathbf{p},j}(\mathbf{p}) = 0$.

Definition 3.1 The *tangent cone* of a set S at $\mathbf{p}$ is the variety

$$C_{\mathbf{p}}(S) = V(f_{\mathbf{p},min} \mid f \in I(S)). \tag{7}$$

Standard bases compute tangent cones. We refer to [2, §9.7] for a careful introduction to the tangent cone. A result of commutative algebra states that the dimension of a variety at a point coincides with the dimension of the tangent cone at that point.

4 Computing Multiplicities

Let $\mathbf{z} \in \mathbb{C}^n$ be an isolated solution of a polynomial system $f(\mathbf{x}) = \mathbf{0}$. The intersection multiplicity $m_{\mathbf{z}}$ is defined algebraically as the dimension of the local quotient ring, formally denoted as

$$m_{\mathbf{z}} = \dim_{\mathbb{C}} \left(\mathbb{C}[\mathbf{x}]_{\langle x_1 - z_1, x_2 - z_2, \ldots, x_n - z_n \rangle} / \langle f \rangle \right). \tag{8}$$

The analogy with global quotient rings should become clear via the example $\langle x^2, xy, y^2 \rangle$, illustrated in Figure 4.

To compute a standard basis for the ideal $I = \langle x^3 - yz, y^3 - xz, z^3 - xy \rangle$, goes like this in Singular, using a negative degree reverse lexicographical ordering over a finite field $\mathbb{Z}_{32003}$:

```
                     SINGULAR                             /
 A Computer Algebra System for Polynomial Computations   /   version 3-0-2
                                                       0<
     by: G.-M. Greuel, G. Pfister, H. Schoenemann        \   July 2006
FB Mathematik der Universitaet, D-67653 Kaiserslautern    \
```

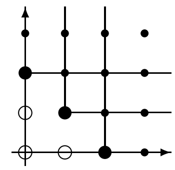

Figure 4: Staircase representation of $\langle x^2, xy, y^2 \rangle$. The big black dots correspond to the generators x^2, xy, and y^2. The monomials under the staircase 1, x, and y represented by the empty circles are a basis for the quotient ring. The small black dots above the staircase are the generated monomials. The multiplicity of $(0,0)$ equals three, the number of monomials under the staircase.

```
> ring R = 32003,(x,y,z),ds;
> poly f1 = x**3-y*z;
> poly f2 = y**3-x*z;
> poly f3 = z**3-x*y;
> ideal i = f1,f2,f3;
> ideal j = std(i);
> j;
j[1]=xy-z3
j[2]=xz-y3
j[3]=yz-x3
j[4]=x4-z4
j[5]=y4-z4
j[6]=z5
```

The output `j` contain the polynomials in the standard basis. Note that `z5` means z^5. If we count the standard monomials (monomials not belonging to the initial ideal), then we should find 11. The Singular session then continues as

```
> mult(j);
11
```

Other measures of the difficulty of a multiple root are the Milnor and Tjurina numbers.

Definition 4.1 The set of *formal power series* over $\mathbb{C}$ is

$$\mathbb{C}[[\mathbf{x}]] = \left\{ \sum_{\mathbf{a} \in \mathbb{Z}^n} c_{\mathbf{a}} \mathbf{x}^a \mid c_{\mathbf{a}} \in \mathbb{C} \right\}. \quad (9)$$

The set of *convergent power series* is

$$\mathbb{C}\{\mathbf{x}\} = \left\{ \sum_{\mathbf{a} \in \mathbb{N}^n} c_{\mathbf{a}} \mathbf{x}^a \mid c_{\mathbf{a}} \in \mathbb{C} \right\}. \quad (10)$$

For a singular point at the origin, *the Milnor number* μ is

$$\mu = \dim_{\mathbb{C}} \mathbb{C}\{\mathbf{x}\} \Big/ \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle . \quad (11)$$

The *Tjurina number* τ is

$$\tau = \dim_{\mathbb{C}} \mathbb{C}[[\mathbf{x}]] \Big/ \left\langle f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle . \quad (12)$$

Intuitively, the higher the Milnor and Tjurina numbers, the more complicated the structure of the singular point is.

We end with a Macaulay 2 session taking the same example as in the Singular session. Starting with the definition of a negative lexicographic order, we compute a standard basis and obtain the multiplicity via the cardinality of the quotient ring.

```
$ M2
Macaulay2, version 1.3.1
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
               PrimaryDecomposition, ReesAlgebra, SchurRings, TangentCone

i1 : R = QQ[x,y,z,MonomialOrder=>{Weights=>{-1,0,0},Weights=>{0,-1,0},
                                   Weights=>{0,0,-1}},Global=>false];

i2 : I = ideal "x3-yz,y3-xz,z3-xy";

o2 : Ideal of R

i3 : I

                     3   3         3
o3 = ideal (- y*z + x , y  - x*z, z  - x*y)

o3 : Ideal of R

i4 : B = gens gb I

o4 = | x5-x6yz x2y-x3y2z x2z-x3yz2 xy2-x3z2 xz2-x3y2 y3-xz yz-x3 z3-xy |

             1       8
o4 : Matrix R  <--- R

i5 : transpose B

o5 = {-8} | x5-x6yz   |
     {-6} | x2y-x3y2z |
     {-6} | x2z-x3yz2 |
     {-5} | xy2-x3z2  |
     {-5} | xz2-x3y2  |
     {-3} | y3-xz     |
     {-3} | yz-x3     |
     {-3} | z3-xy     |

             8       1
o5 : Matrix R  <--- R

i6 : S = R/I;

i7 : basis(S)

o7 = | 1 x x2 x3 x4 xy xz y y2 z z2 |

             1       11
o7 : Matrix S  <--- S
```

5 Exercises

1. Consider $I = \langle x_1^2 + x_1^3, x_2^2 \rangle$.
   (a) Compute the multiplicity of $(0,0)$ by adding a small ϵ to both equations. Explain how to derive the multiplicity by looking at the perturbed ideal. What are the conditions for making this calculation work?
   (b) Verify the multiplicity with Singular.

2. Take $I = \langle f_1 = x_1^3 + x_1x_2^2, f_2 = x_1x_2^2 + x_2^3, f_3 = x_1^2x_2 + x_1x_2^2 \rangle$. Use Singular to compute a standard basis for I. Choose at least two different orderings which lead to different standard bases. Observe that the number of standard monomials is the same for both bases.

3. Download the software from `http://www.oliverlabs.net`. Use the Cayley cubic as a benchmark to compare the visualization capabilities of Oliver's software with what Maple and/or Sage has to offer.

4. The classification of hypersurface singularities has led to special names [4, Example 3.4.25]:
   - A_k: $f(x_1, x_2, x_3) = x_1^{k+1} + x_2^2 + x_3^2$, $k \geq 1$;
   - D_k: $f(x_1, x_2, x_3) = x_1x_2^2 + x_1^{k-1} + x_3^2$, $k \geq 4$;
   - E_6: $f(x_1, x_2, x_3) = x_1^3 + x_2^4 + x_3^2$;
   - E_7: $f(x_1, x_2, x_3) = x_1^3 + x_1x_2^3 + x_3^2$;
   - E_8: $f(x_1, x_2, x_3) = x_1^3 + x_2^5 + x_3^2$.

   Use software (see the previous exercise) to visualize several instances of these singularities.

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