Numerical Schubert Calculus

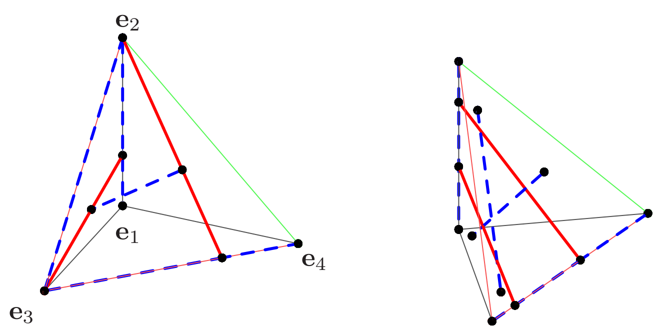
Homotopy continuation methods are enumerative methods as they are efficient tools to enumerate all solutions. Enumerative geometry is also known as Schubert calculus, whence Numerical Schubert Calculus [2] covers numerical methods to solve problems in enumerative geometry.

1 Four Lines in 3-Space

A classical problem in geometry goes as follows. Given four lines in space, in general position with respect to each other. How many lines meet all four given lines?

To answer this question, we identify coordinates $[x_0 : x_1 : x_2 : x_3] \in \mathbb{P}^3$ in projective 3-space with ordinary points $(x_0, x_1, x_2, x_3) \in \mathbb{C}^4$. As any line is spanned by two points, we give the generators for the line as two points in $\mathbb{C}^4$. For two distinct points $\mathbf{a}, \mathbf{b} \in \mathbb{C}^4$, we denote the line they span by $\langle \mathbf{a}, \mathbf{b} \rangle$. The columns of the 4-by-4 identity matrix span the natural standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ for $\mathbb{C}^4$. Without loss of generality we may assume that the first two of the four given lines are spanned as $L_1 = \langle \mathbf{e}_1, \mathbf{e}_2 \rangle$ and $L_2 = \langle \mathbf{e}_3, \mathbf{e}_4 \rangle$.

The picture below shows at the left a special configuration of the four given lines. The configuration is special in the sense that the third line is given as $L_3 = \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$. At the right, the third line is moved in general position.



If $L_1 = \langle \mathbf{e}_1, \mathbf{e}_2 \rangle$ and $L_2 = \langle \mathbf{e}_3, \mathbf{e}_4 \rangle$, then take X as spanned by points on L_1 and L_2. Since the columns in X are determined up to a nonzero constant, we may fix the topmost nonzero entries to one. Then we are left with two indeterminates: x_{12} and x_{24}. The two intersection conditions imposed by L_3 and L_4 will determine x_{12} and x_{24}. Associated to X is the special line $S_{[2\ 4]}$

$$X = \begin{bmatrix} 1 & 0 \\ \star & 0 \\ 0 & 1 \\ 0 & \star \end{bmatrix} \quad S_{[2\ 4]} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \langle \mathbf{e}_1, \mathbf{e}_3 \rangle \tag{1}$$

Observe

$$\det([X | S_{[2\ 4]}]) = 0 \quad \Leftrightarrow \quad x_{12} = 0 \text{ or } x_{24} = 0 \tag{2}$$

When $x_{12} = 0$, the other variable x_{24} is free and we use this freedom to satisfy the fourth intersection condition: $\det([X|L_4) = 0$.

With a Pieri homotopy, we move the special line into general position:

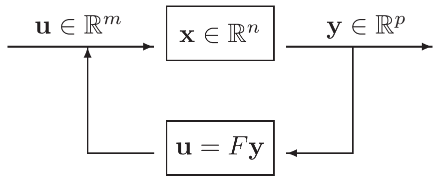
$$h(X, t) = \left\{ \begin{array}{r} \det([X|(1-t)S_{[2\ 4]} + tL_3]) = 0 \\ \det([X|L_4]) = 0. \end{array} \right. \tag{3}$$

As t moves from 0 to 1, the third line moves from its special position at $S_{[2\ 4]}$ to a given general position at L_3. Cheater's homotopy can then be used to solve any given problem with specific real input data.

2 Static Output Pole Placement

This section is an abbreviation of a section in [7] where we rephrase the static pole placement problem into a problem of enumerative geometry [1].

Suppose we want to control a plant with n internal states $\mathbf{x} \in \mathbb{R}^n$ that takes m-inputs $\mathbf{u} \in \mathbb{R}^m$ and produces p-outputs $\mathbf{y} \in \mathbb{R}^p$ with a static compensator, given by the feedback law F, and applied as $\mathbf{u} = F\mathbf{y}$. Schematically we picture this situation in the figure below.



In general we consider a system with input $\mathbf{u} \in \mathbb{R}^m$, output $\mathbf{y} \in \mathbb{R}^p$, and internal states $\mathbf{x} \in \mathbb{R}^n$, $n = m+p$, whose evolution in time t is governed by the first-order linear differential equation

$$\left\{ \begin{array}{ll} \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), & A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, \\ \mathbf{y}(t) = C\mathbf{x}(t), & C \in \mathbb{R}^{p \times n}, \\ \mathbf{u}(t) = F\mathbf{y}(t), & F \in \mathbb{R}^{m \times p}, \end{array} \right. \tag{4}$$

where the last part expresses the control of the input by constant output feedback F.

Substitution into the first equation of (4) yields $\dot{\mathbf{x}}(t) = (A + BFC)\mathbf{x}(t)$, whose characteristic polynomial is

$$\varphi(s) = \det(sI_n - A - BFC). \tag{5}$$

The roots s_i: $\varphi(s_i) = 0$, for $i = 1, 2, \ldots, n$, are the natural frequencies of the controlled system.

The pole placement problem is an inverse problem: given A, B, C, and φ (determined by the frequencies s_i), compute the feedback laws F that satisfy $\varphi(s_i) = 0$, for $i = 1, 2, \ldots, n$.

We rewrite the expression (5) for $\varphi(s)$ as follows:

$$\det(sI_n - A - BFC) = 0 \quad \Leftrightarrow \quad \det \left[\begin{array}{ccc} sI_n - A & 0 & -B \\ -C & I_p & 0 \\ 0 & -F & I_m \end{array} \right] = 0 \tag{6}$$

$$\Leftrightarrow \quad \det \left[\left(\begin{array}{ccc} 0 & -F & I_m \end{array} \right) \left(\begin{array}{c} X(s) \\ Y(s) \\ U(s) \end{array} \right) \right] = 0 \tag{7}$$

$$\Leftrightarrow \quad \det \left[\begin{array}{cc} U(s) & F \\ Y(s) & I_m \end{array} \right] = 0 \tag{8}$$

where I_p and I_m are the respective p-by-p and m-by-m identity matrices. The first step in (6) uses a determinantal identity. Secondly, we apply the fact that

$$\det \left[\begin{array}{c} P(s) \\ M(s) \end{array} \right] = c \, \det \left[M(s)Q(s) \right], \quad c \in \mathbb{R}, \ c \neq 0, \tag{9}$$

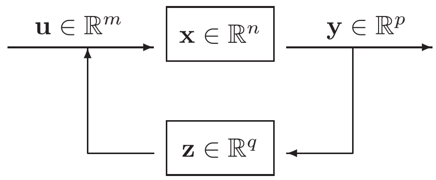
for

$$P(s) = \left[\begin{array}{ccc} sI_n - A & 0 & -B \\ -C & I_p & 0 \end{array} \right], \quad Q(s) = \left[\begin{array}{c} X(s) \\ Y(s) \\ U(s) \end{array} \right] \quad \text{and} \quad P(s)Q(s) = 0. \tag{10}$$

The $(n + p + m)$-by-m polynomial matrix $Q(s)$ describes the the behavior of the system explicitly, i.e.: it gives for an m-input the new states, output and feedback. The third step is a simple elaboration to transform the equations of the pole placement problem into the familiar geometric form (8) we have used to solve it.

3 Dynamic Output Pole Placement

This section is an abbreviation of a section in [3] where we rephrase the dynamic pole placement problem into a problem of enumerative geometry [6], [5]. Suppose we want to control a plant with n internal states $\mathbf{x} \in \mathbb{R}^n$ that takes m-inputs $\mathbf{u} \in \mathbb{R}^m$ and produces p-outputs $\mathbf{y} \in \mathbb{R}^p$ with a dynamic compensator that has q internal states $\mathbf{z} \in \mathbb{R}^q$. Schematically we picture this situation in the figure below.



The evolution in time of the plant is described by a system of first-order differential equations:

$$\left\{ \begin{array}{ll} \dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} & \text{with } \mathbf{x} \in \mathbb{R}^n, \mathbf{u} \in \mathbb{R}^m, \mathbf{y} \in \mathbb{R}^p, \\ \mathbf{y} = C\mathbf{x} & \text{and } A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}. \end{array} \right. \tag{11}$$

The dynamic compensator obeys a qth order differential equation, described by the system:

$$\left\{ \begin{array}{l} \dot{\mathbf{z}} = F\mathbf{z} + G\mathbf{y} \\ \mathbf{u} = H\mathbf{z} + K\mathbf{y} \end{array} \right. \quad \text{with} \quad \mathbf{z} \in \mathbb{R}^q \quad \text{and} \quad \begin{array}{l} F \in \mathbb{R}^{q\times q}, G \in \mathbb{R}^{q\times p}, \\ H \in \mathbb{R}^{m\times q}, K \in \mathbb{R}^{m\times p}. \end{array} \tag{12}$$

After elimination of $\mathbf{u}$ and $\mathbf{y}$ concatenation of (11) and (12) yields the following closed-loop system:

$$\left[\begin{array}{c} \dot{\mathbf{x}} \\ \dot{\mathbf{z}} \end{array} \right] = \left[\begin{array}{cc} A + BKC & BH \\ GC & F \end{array} \right] \left[\begin{array}{c} \mathbf{x} \\ \mathbf{z} \end{array} \right] \tag{13}$$

The behavior of this closed-loop system is determined by the $n+q$ eigenvalues of the matrix in (13). For a plant given by the matrix triplet (A, B, C) and $n+q$ eigenvalues, the dynamic pole placement problem asks for the matrix quadruples (F, G, H, K) which determine the dynamic compensators that yield closed-loop systems with a specific set of eigenvalues.

The dynamic pole placement problem is a geometric problem. In rewriting the characteristic equation of (13) the subscripts in I denote the dimension of the identity matrix.

$$\det \left(s \left[\begin{array}{cc} I_n & 0 \\ 0 & I_q \end{array} \right] - \left[\begin{array}{cc} A + BKC & BH \\ GC & F \end{array} \right] \right) = 0 \tag{14}$$

$$\Leftrightarrow \quad \det \left[\begin{array}{cc} sI_n - A - BKC & -BH \\ -GC & sI_q - F \end{array} \right] = 0 \quad \Leftrightarrow \quad \textit{omitted} \ldots \tag{15}$$

$$\Leftrightarrow \quad \det \left[\begin{array}{cc} I_p & C(sI_n - A)^{-1}B \\ H(sI_q - F)^{-1}G + K & I_m \end{array} \right] = 0 \tag{16}$$

The determinant in (16) represents the intersection of m-planes defined by the given triplet (A, B, C) with p-planes determined by the unknown quadruple (F, G, H, K). As these p-planes depend on the variable s introduced by the the term $(sI_q - F)^{-1}$, they have maximal minors of degree q and we call them degree q-maps. Thus the dynamic pole placement problem is equivalent to the computation of all degree q-maps into the Grassmannian of p-planes that meet n given m-planes at prescribed s-values. It is easy to see that for each specification of an eigenvalue λ_i the condition that the characteristic polynomial (16) vanish at $s = \lambda_i$ enforces one polynomial condition on the set of degree q-maps.

Note that when $q = 0$ we are solving the static pole placement problem ($\mathbf{u} = K\mathbf{y}$ and $F, G, H = 0$) and are looking for maps of degree 0 (i.e. constant maps) which meet a specific set of given m-planes. In this case the characteristic equation (16) has degree n and we can find solution planes whenever n is less than the dimension mp of the space of p-planes in $(m+p)$-dimensional space. At the critical dimension $n = mp$, we expect the number of such solutions to be finite.

4 Pieri Homotopies

The problem specification for Pieri Homotopies is

Input: mp m-planes in $\mathbb{C}^{m+p}$, $L_1, L_2, \ldots, L_{mp}$.
Output: p-planes X : $X \cap L_i \neq \{\mathbf{0}\}$, $i = 1, 2, \ldots, mp$.

For the problem of four given lines in 3-space we have $m = 2$ and $p = 2$. The key data structure is *a localization pattern* to represent the solution X:

$$X = \left[\begin{array}{cc} 1 & 0 \\ \star & 1 \\ \star & \star \\ 0 & \star \end{array} \right] \star \neq 0, \qquad \begin{array}{c} \det([X|L_i]) = 0, \\ i = 1, 2, 3, 4. \end{array} \tag{17}$$

Four general lines L_i will determine X uniquely.

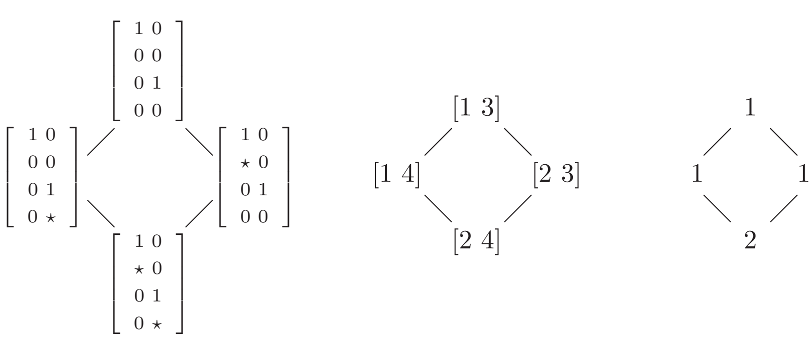
A *localization pattern* is an element of $\{0, \star\}^{(m+p)\times p}$ such that all stars in a column are contiguous and the row indices in which the bottommost and topmost stars occur strictly increase as functions of the column number. These row indices are called the *top* and *bottom pivots* respectively. A p-plane *fits a localization pattern* if it can be represented by a matrix of generators with zero entries everywhere the localization pattern prescribes them.

Consider for example the case where $p = 2, m = 4$ and we are looking for all 2-planes which meet eight 4-planes. We can take the last two planes as $L_7 = \langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 \rangle$ and $L_8 = \langle \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6 \rangle$. For any 2-plane that meets both these planes nontrivially we may choose its generators such that one generator lies in L_7 and the other in L_8. Such a 2-plane is represented by a variable matrix

$$\left[\begin{array}{cc} x_{11} & 0 \\ x_{21} & 0 \\ x_{31} & x_{32} \\ x_{41} & x_{42} \\ 0 & x_{52} \\ 0 & x_{62} \end{array} \right] \text{ which fits the pattern } \left[\begin{array}{cc} \star & 0 \\ \star & 0 \\ \star & \star \\ \star & \star \\ 0 & \star \\ 0 & \star \end{array} \right]. \tag{18}$$

Planes which satisfy this localization pattern already meet L_7 and L_8.

We specialize the input planes so the intersection condition with the input plane is special position is achieved setting pivot elements in the solution plane to zero. Given a localization pattern X, *the special m-plane for the top (bottom) pivots of* X is the m-plane S_X spanned by the standard basis vectors not indexed by top (bottom) pivots. Solution p-planes which meet fit a localization pattern X and meet its special m-plane S_X must fit a set of localization patterns pattern obtained by turning a topmost (bottommost) star of X to a zero. For patterns X and Y, if Y is obtained from X by setting one of the pivots of X to zero, we say that Y *is a child of* X. With localization posets, we obtain root counts:



Lemma 4.1 (Hypersurface Pieri rule) *Let S_X be the special m-plane for top (bottom) pivots of the localization pattern X. Every p-plane that fits X and meets S_X also fits one of the (at most p) children of X. Conversely every p-plane that fits a child of X fits X and meets S_X.*

Proof. A p-plane that fits X meets S_X if, and only if, for all specific values for the variables corresponding to the stars in the pattern X, we can find a linear combination of the columns of the p-plane which lies in S_X. This means that the intersection of the p-plane with S_X can be represented as a linear combination of the columns of S_X. Finding this linear combination is equivalent to finding a nontrivial element of the kernel of $[X|S_X]$. Since the m columns of S_X have pivots distinct from those of the p columns of X, the matrix $[X|S_X]$ is an $(m+p)\times(m+p)$-matrix with $m+p$ distinct pivot rows. So we can rearrange $[X|S_X]$ into a triangular matrix, which is singular precisely when one of the diagonal elements coming from X is specialized to zero. □

Bringing the input plane L into special position S_X is done via a Pieri homotopy deforming L into S_X as t goes from 1 to 0. Solutions to the specialized problem have zeroes at the pivot positions of the pattern X and are start solutions at $t=0$.

Definition 4.1 Let X be a localization pattern and S_X the special m-plane for top (bottom) pivots of X. Suppose that $n-1$ general intersection conditions are satisfied for children Y of X i.e.: $\det(Y|L_i)=0$, for $i=1,2,\ldots,n-1$. To satisfy the nth intersection condition, the *Pieri homotopy* $H(X,t)=\mathbf{0}$ is:

$$H(X,t) = \left\{ \begin{array}{rl} \det(X|(1-t)S_X + tL_n) = 0 & \\ \det(X|L_i) = 0 & i=1,2,\ldots,n-1 \end{array} \right. \quad t\in[0,1]. \tag{19}$$

Lemma 4.2 (regularity of Pieri homotopy) *Consider a localization pattern X with $p+n$ stars and n complex m-planes L_i in general position. Suppose we are given all p-planes that meet L_i, $i=1,2,\ldots,n-1$ and fit one of the children of X. Then the Pieri homotopy defines regular paths of p-planes, that start at the given p-planes and end at those p-planes which meet all n general m-planes L_i while fitting X.*

Proof. Our working space is a product of projective spaces, with as many spaces as there are columns in X. To describe this multi-projective space more precisely, we count the stars in X. For X defined by p-tuples of top and bottom pivots, respectively $\alpha,\beta\in\mathbb{N}^p$, we embed the p-planes into $\mathbb{P}^{d_1}\times\mathbb{P}^{d_2}\times\cdots\times\mathbb{P}^{d_p}$ with $d_i=\beta_i-\alpha_i$, $i=1,2,\ldots,p$. To fix an affine coordinate chart we set one coordinate to 1 in every column by scaling the corresponding generator of the p-plane.

Essentially we are applying a multi-homogeneous homotopy in a general situation. By the assumption of this Lemma, all p-planes at $t=0$ are regular solutions, as the m-planes L_i are in general position. This general position is maintained for all $t\in[0,1]$, ensuring the smoothness of the solution paths.

We still have to show that we will find all p-planes. Suppose that at $t=1$ there are more solutions than the number of paths we started with. Going backwards with those additional solutions from $t=1$ to $t=0$ we either move to a singular solution, or to a solution at infinity. But at $t=0$, all solutions are regular. By Lemma 4.1, all solutions have been found and since the localization patterns admit any affine chart, there are no solutions at infinity for $t=0$ □

Application of Lemma 4.1 and Lemma 4.2 inductively justifies the combinatorial root count derived from the poset of localization patterns.

Theorem 4.1 *Consider a localization pattern X with $p+n$ stars. For $n=0$, X counts for one solution. For $n>0$, the number of p-planes fitting X and meeting n general m-planes equals the sum of the number of solution planes fitting the children of X and meeting $n-1$ general m-planes.*

5 Curves producing Planes

In this section we extend the classical problem of 4 lines in 3-space considering a curve that returns lines when evaluated. As usual, we identify coordinates $[x_0 : x_1 : x_2 : x_3] \in \mathbb{P}^3$ in projective 3-space with ordinary points $(x_0, x_1, x_2, x_3) \in \mathbb{C}^4$. A line L spanned by two distinct points $\mathbf{a}, \mathbf{b} \in \mathbb{C}^4$, is denoted by $L = \langle \mathbf{a}, \mathbf{b} \rangle$. Often we identify L with a matrix whose columns contain the generators of the line.

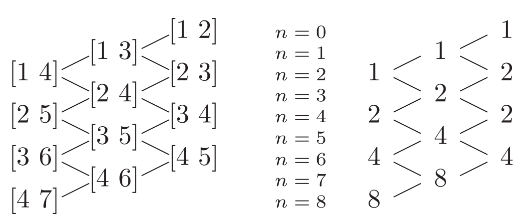
Denoting a nonzero element by $\star$, the representation we use for line producing curves $X(s)$ is

$$X(s) = \left[\begin{array}{cc} 1 & 0 \\ \star & 1 \\ \star & \star \\ \star & \star \end{array} \right] + \left[\begin{array}{cc} 0 & \star \\ 0 & \star \\ 0 & \star \\ 0 & 0 \end{array} \right] s. \qquad (20)$$

The rather particular placement of the stars in the representation of $X(s)$ is needed for the root count. If we stack the two matrices in $X(s)$ on top of each other, the stars are aligned contiguously and we may use bottom pivots as the shorthand notation:

$$X(s) \quad \Leftrightarrow \quad \left[\begin{array}{cc} 1 & 0 \\ \star & 1 \\ \star & \star \\ \star & \star \\ 0 & \star \\ 0 & \star \\ 0 & \star \\ 0 & 0 \end{array} \right] \quad \Leftrightarrow \quad [4\ 7]. \qquad (21)$$

The formal root count shows there are 8 curves which meet 8 given lines at specified interpolation points:



After the initial level ($n = 0$), we see eight levels in the poset of brackets. At each level we satisfy one intersection condition. The poset serves as a blue print for the sequence of Pieri homotopies executed to solve this intersection problem.

6 Quantum Pieri Homotopies

To solve the dynamic output pole placement problem, we consider

Input: $n = mp + q(m + p)$ general m-planes in $\mathbb{C}^{m+p}$, $L_1, L_2, \ldots, L_n$, sampled at n interpolation points, $s_1, s_2, \ldots s_n$.

Output: maps $X(s)$ of degree q producing p-planes: $X(s_i) \cap L_i \neq \{\mathbf{0}\}$, $i = 1, 2, \ldots, n$.

For $X(s)$ to meet L_n at s_n, we use the Pieri homotopy. But first, we must compactify the curves:

$$X(s,t) = \begin{bmatrix} 1 & 0 \\ x_{21}^{(0)} & 1 \\ x_{31}^{(0)} & x_{32}^{(0)} \\ x_{41}^{(0)} & x_{42}^{(0)} \end{bmatrix} t + \begin{bmatrix} 0 & x_{12}^{(1)} \\ 0 & x_{22}^{(1)} \\ 0 & x_{23}^{(1)} \\ 0 & 0 \end{bmatrix} s = \begin{bmatrix} t & x_{12}^{(1)} s \\ x_{21}^{(0)} t & t + x_{22}^{(1)} s \\ x_{31}^{(0)} t & x_{32}^{(0)} t + x_{22}^{(1)} s \\ x_{41}^{(0)} t & x_{42}^{(0)} t \end{bmatrix}. \tag{22}$$

As $t \rightarrow 0$, the highest order terms in $X(s,t)$ become dominant and we can apply the same logic as before, choosing again special planes to meet $X(s,t)$.

The quantum Pieri homotopies then take the following shape:

$$h(X(s,t), s, t) = \left\{ \begin{array}{r} \det(\ X(s,t)\ |\ (1-t)S_X + tL_n\ ) = 0 \\ (s-1)(1-t) + (s - s_n)t = 0 \\ \det(\ X(s_i, t_i)\ |\ L_i\ ) = 0 \\ i = 1, 2, \ldots, n-1 \end{array} \right. \quad t \in [0,1].$$

As t goes from 0 to 1, the interpolation point goes from $(1,0)$ to $(s_n, 1)$, from ∞ to s_n.

The "quantum" in the section title suggests a connection to physics. We refer to [4] for the connections between enumerative geometry and string theory.

The module `schubert` of the `phcpy` package offers an implementation of the quantum Pieri homotopies.

```
>>> from phcpy import schubert
>>> (m, p, q) = (2, 2, 1)
>>> schubert.pieri_root_count(m, p, q)
Pieri root count for (2, 2, 1) is 8
the localization poset :
n = 0 : ([3 4],[3 4],1)([2 5],[2 5],1)
n = 1 :
n = 2 : ([2 4],[3 5],2)
n = 3 :
n = 4 : ([2 3],[3 6],2)([2 3],[4 5],2)([1 4],[3 6],2)([1 4],[4 5],2)
n = 5 :
n = 6 : ([1 3],[4 6],8)
n = 7 :
n = 8 : ([1 2],[4 7],8)
>>> dim = m*p + q*(m+p)
>>> planes = [schubert.random_complex_matrix(m+p, m) for k in range(0, dim)]
>>> points = schubert.random_complex_matrix(dim, 1)
>>> (system, sols) = schubert.run_pieri_homotopies(m, p, q, planes, points)
>>> schubert.verify(system, sols)
(-6.41153796721e-14-3.11417558407e-14j)
(5.44009282066e-15+1.02140518266e-14j)
(-2.01505478969e-14+2.94209101526e-15j)
(-1.83186799063e-14-6.82787160144e-15j)
(4.16333634234e-15-8.72080185843e-14j)
(1.94844140822e-14-7.49955653134e-14j)
(-6.61137811164e-14+2.39253061807e-14j)
(8.86513085163e-14+1.60427227058e-14j)
the total check sum : (-5.09592368303e-14-1.47049039612e-13j)
```

7 Exercises

1. In $\mathbb{C}^6$, there are 462 3-planes which meet 12 given 4-planes in general position. Create the poset of brackets to compute this combinatorial root count.

2. Use `phc -e` or `phcpy.schubert` to solve a particular instance of the 462 3-planes meeting 12 given 4-planes in general position in $\mathbb{C}^6$.

3. Consider the case of curves of degree 1 producing 2-planes meeting 3-planes in $\mathbb{C}^5$. How many 3-planes (and corresponding interpolation points) must one give to fully specify the curves? How many such curves are there?

4. Use `phc -e` or `phcpy.schubert` to compute the root count for the problem considered in exercise 1.

References

[1] R.W. Brockett and C.I. Byrnes. Multivariate Nyquist criteria, root loci, and pole placement: a geometric viewpoint. *IEEE Trans. Automat. Control*, 26:271–284, 1981.

[2] B. Huber, F. Sottile, and B. Sturmfels. Numerical Schubert calculus. *J. Symbolic Computation*, 26(6):767–788, 1998.

[3] B. Huber and J. Verschelde. Pieri homotopies for problems in enumerative geometry applied to pole placement in linear systems control. *SIAM J. Control Optim.*, 38(4):1265–1287, 2000.

[4] S. Katz. *Enumerative Geometry and String Theory*, volume 32 of *Student Mathematical Library. IAS/Park City Mathematical Subseries*. AMS, 2006.

[5] M.S. Ravi, J. Rosenthal, and X. Wang. Dynamic pole placement assignment and Schubert calculus. *SIAM J. Control and Optimization*, 34(3):813–832, 1996.

[6] J. Rosenthal. On dynamic feedback compensation and compactifications of systems. *SIAM J. Control and Optimization*, 32(1):279–296, 1994.

[7] J. Verschelde. Numerical evidence for a conjecture in real algebraic geometry. *Experimental Mathematics*, 9(2):183–196, 2000.