Binomial Ideals

Binomial ideals offer an interesting class of examples. Because they occur so frequently in various applications, the development methods for binomial ideals is justified.

1 Binomial Ideals

Consider $\mathbb{K}[\mathbf{x}]$, with $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$. Typically we will assume that $\mathbb{K}$ is algebraically closed, so $\mathbb{K} = \mathbb{C}$ is our default coefficient field. Then

$$I = \left\langle \ c_{\mathbf{a}} x^{\mathbf{a}} - c_{\mathbf{b}} x^{\mathbf{b}} \mid \mathbf{a}, \mathbf{b} \in \mathbb{N}^n, c_{\mathbf{a}}, c_{\mathbf{b}} \in \mathbb{K}^* \ \right\rangle \tag{1}$$

is a binomial ideal. A polynomial is a binomial if it has exactly two monomials with a nonzero coefficient. A binomial ideal is generated by binomials. A *pure difference ideal* is an ideal generated by differences of monic monomials, i.e.: all generators are of the form $\mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{b}}$. The following proposition contains an algorithm to solve a zero dimensional pure difference ideal.

Proposition 1.1 *Let I be a zero dimensional pure difference ideal. There is a primitive root of unity ξ, such that all complex solutions of I are contained in the cyclotomic field $\mathbb{Q}(\xi)$.*

Proof. Let $\mathcal{G}$ be a lexicographic Gröbner basis. Because all S-polynomials are pure difference binomials, $\mathcal{G}$ consists of pure difference binomials. As the ideal is zero dimensional and because a lexicographic order eliminates, at least one of the binomials in $\mathcal{G}$ is univariate.

The solutions of the univariable equations exists in a cyclotomic field. By substituting the solution for that variable in the other equations, an univariate equation in another variable is obtained. After extending the partial solutions, all roots of unity encountered during univariate solving define $\mathbb{Q}(\xi)$ where the solutions live. □

If the generators are of the form $\mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{b}}$ (all coefficients $c_{\mathbf{a}}$ and $c_{\mathbf{b}}$ are one), then the binomial ideal is a toric ideal. Because the exponents determine the structure of the ideal, we then define a toric ideal as

$$I_A = \left\langle \ \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \mid \mathbf{u}, \mathbf{v} \in \mathbb{N}^n \text{ and } A\mathbf{u} = A\mathbf{v} \ \right\rangle. \tag{2}$$

The solution set of a toric ideal is a toric variety — see [6] for a short and [1] for a longer definition. As an alternative to the ideal description, a toric variety over $\mathbb{C}$ is defined as a complex algebraic variety with an action of $(\mathbb{C}^*)^n$ and a dense open subset isomorphic to $(\mathbb{C}^*)^n$ carrying the regular action, i.e.: a toric variety is an algebraic torus closure.

The efficient manipulation of monomial ideals requires combinatorics [7]. Via the introduction of new variables we can rewrite any system of polynomials into a system of trinomials, of polynomials with three monomials (or less). This trick does not allow the reduction of any system to a binomial system. Binomial ideals have special properties, for instance:

Theorem 1.1 (Theorem 2.6 in [2]) *If $\mathbb{K}$ is algebraically closed and I is a binomial ideal in $\mathbb{K}[\mathbf{x}]$, then every associated prime of I is generated by binomials.*

The condition that $\mathbb{K}$ is algebraically closed is essential, as the following example over $\mathbb{Q}$ shows: $\langle x^3 - 1 \rangle = \langle x - 1 \rangle \cap \langle x^2 + x + 1 \rangle$. If we extend $\mathbb{Q}$ with $w = e^{(2\pi\sqrt{-1})/3}$, then over $\mathbb{Q}(w)$ the binomial ideal factors as $\langle x^3 - 1 \rangle = \langle x - 1 \rangle \cap \langle x + (1 - \sqrt{-3})/2 \rangle \cap \langle x + (1 + \sqrt{-3})/2 \rangle$.

The frequent occurring of binomial ideals in applications justifies the development of specific methods to solve binomial systems.

In polyhedral homotopies we encounter the special case where the number of generators equals n and only solutions in $(\mathbb{C}^*)^n$ are interesting. After writing the system in a normal form as $\mathbf{x}^A = \mathbf{c}$, with $A \in \mathbb{Z}^{n \times n}$ and $\mathbf{c} \in (\mathbb{C}^*)^n$, we conclude that there are exactly as many regular solutions as $|\det(A)|$.

2 Commuting Birth-and-Death Processes

In [3], the problem of commuting birth-and-death processes is investigated as an application of combinatorics and algebraic statistics. In models arising in ecology and queuing theory one studies population sizes and numbers of individual waiting in a queue. Markov chains are described by tridiagonal transition matrices P, $P(i,j)$ is the probability of going from step i to j. In a higher-dimensional model the state space is a product of intervals in higher-dimensional lattices, e.g.: ecology: keep track of the type of individuals in a population; queuing: several servers have each their own set of customers. The mathematical tools in one dimension are orthogonal polynomials. In higher dimension, we compute a binomial primary decomposition.

Consider an example in dimension two. Define a grid $E = \{0,1,\ldots,m\} \times \{0,1,\ldots,n\}$ where (i,j) is connected to $(k,\ell) \Leftrightarrow |i-k| + |j-\ell| = 1$. The transition probabilities are

$$\begin{aligned}
\text{go left: } L_{i,j} &= \text{prob}\{Z_{k+1} = (i-1,j) \mid Z_k = (i,j)\} && (3)\\
\text{go right: } R_{i,j} &= \text{prob}\{Z_{k+1} = (i+1,j) \mid Z_k = (i,j)\} && (4)\\
\text{go down: } D_{i,j} &= \text{prob}\{Z_{k+1} = (i,j-1) \mid Z_k = (i,j)\} && (5)\\
\text{go up: } U_{i,j} &= \text{prob}\{Z_{k+1} = (i,j+1) \mid Z_k = (i,j)\} && (6)
\end{aligned}$$

Commuting relations:

$$\begin{aligned}
U_{i,j}R_{i,j+1} &= R_{i,j}U_{i+1,j} && \text{(up-right)} && (7)\\
D_{i,j+1}R_{i,j} &= R_{i,j+1}D_{i+1,j+1} && \text{(down-right)} && (8)\\
D_{i+1,j+1}L_{i+1,j} &= L_{i+1,j+1}D_{i,j+1} && \text{(down-left)} && (9)\\
U_{i+1,j}L_{i+1,j+1} &= L_{i+1,j}U_{i,j} && \text{(up-left)} && (10)
\end{aligned}$$

The commuting relations define a system of quadratic polynomials. Consider in general, the m-dimensional grid: $E = \{0,1,\ldots,n_1\} \times \{0,1,\ldots,n_2\} \times \ldots \times \{0,1,\ldots,n_m\}$. For all pairs (i,j): $1 \leq i < j \leq m$, the commuting requirement

$$\begin{array}{l}
P(u,u+e_i)P(u+e_i,u+e_i+e_j) - P(u,u+e_j)P(u+e_j,u+e_i+e_j),\\
P(u,u+e_i)P(u+e_i,u+e_i-e_j) - P(u,u-e_j)P(u+e_j,u+e_i-e_j),\\
P(u,u-e_i)P(u-e_i,u-e_i+e_j) - P(u,u+e_j)P(u+e_j,u-e_i+e_j),\\
P(u,u-e_i)P(u-e_i,u-e_i-e_j) - P(u,u-e_j)P(u+e_j,u-e_i-e_j)
\end{array} \qquad (11)$$

is a system of quadratic polynomials in the unknowns $P(u,v)$.

The ideal of commuting birth-and-death processes is denoted by $I^{(n_1,n_2,\ldots,n_m)}$, as the ideal generated by the quadratic polynomials in the commuting requirement. In the two dimensional case, $I^{(m,n)}$ is generated by $4mn$ quadratic binomials, for (i,j): $0 \leq i < m$ and $0 \leq j < n$:

$$\begin{aligned}
U_{i,j}R_{i,j+1} - R_{i,j}U_{i+1,j} &= 0, && (12)\\
R_{i,j+1}D_{i+1,j+1} - D_{i,j+1}R_{i,j} &= 0, && (13)\\
D_{i+1,j+1}L_{i+1,j} - L_{i+1,j+1}D_{i,j+1} &= 0, && (14)\\
L_{i+1,j}U_{i,j} - U_{i+1,j}L_{i+1,j+1} &= 0. && (15)
\end{aligned}$$

The smallest example in this family is listed below. The possibilities that

$$\left(\begin{array}{cccc}
0 & 0 & R_{0,0} & 0\\
0 & 0 & 0 & R_{0,1}\\
L_{1,0} & 0 & 0 & 0\\
0 & L_{1,1} & 0 & 0
\end{array}\right) \text{ and } \left(\begin{array}{cccc}
0 & U_{0,0} & 0 & 0\\
D_{0,1} & 0 & 0 & 0\\
0 & 0 & 0 & U_{1,0}\\
0 & 0 & D_{1,1} & 0
\end{array}\right) \qquad (16)$$

commute are revealed by the primary decomposition of
$I^{(1,1)} = \langle U_{0,0}R_{0,1} - R_{0,0}U_{1,0}, R_{0,1}D_{1,1} - D_{0,1}R_{0,0}, D_{1,1}L_{1,0} - L_{1,1}D_{0,1}, L_{1,0}U_{0,0} - U_{1,0}L_{1,1}\rangle$.

3 Cellular decompositions

Following [4], consider $\mathcal{E} \subseteq \{1, 2, \ldots, n\}$ and denote the algebraic torus corresponding to $\mathcal{E}$ by

$$(\mathbb{K}^*)^{\mathcal{E}} = \{ \mathbf{x} \in \mathbb{K}^n \mid x_i \neq 0 \text{ for } i \in \mathcal{E} \text{ and } x_j = 0 \text{ for } j \notin \mathcal{E} \}. \tag{17}$$

The central definition is

Definition 3.1 A proper binomial ideal I in $\mathbb{K}[\mathbf{x}]$ is *cellular* if each variable x_i is either a nonzerodivisor or nilpotent modulo I.

Primary ideals I are cellular as every element in $\mathbb{K}[\mathbf{x}]/I$ is either nilpotent or a nonzerodivisor.

We have a characterization for an ideal I being cellular in the following lemma.

Lemma 3.1 *A proper binomial ideal I in $\mathbb{K}[\mathbf{x}]$ is cellular if and only if there exists a set $\mathcal{E} \subseteq \{1, 2, \ldots, n\}$ of indices of variables in $\mathbf{x}$ such that*

1. $I = \left(I : \left(\prod_{i \in \mathcal{E}} x_i \right)^\infty \right)$*; and*
2. *For every $i \notin \mathcal{E}$, there exists an integer $d_i \geq 0$ such that $\langle x_i^{d_i} \mid i \notin \mathcal{E} \rangle$ is contained in I.*

The `Binomials` package in Macaulay 2 provides an implementation of the following recursive algorithm:

Algorithm 3.1 (cellular decomposition)
Input: a binomial ideal I.
Output: a cellular decomposition of I.

1. If I is cellular, then return I.
2. Choose x_i that is a zerodivisor but not nilpotent modulo I.
3. Determine the power m such that $(I : x_i^m) = (I : x_i^\infty)$.
4. Call the algorithm on $(I : x_i^m)$ and $I + \langle x_i^m \rangle$.

The algorithm to compute a cellular decomposition is the first step in the following algorithm to solve toric ideals.

Algorithm 3.2 (solve toric ideals)
Input: a zero dimensional toric ideal I.
Output: roots of unity to extend $\mathbb{Q}$ and solutions in $V(I)$.

1. Compute a cellular decomposition of I.
2. For each cellular component do

2.1 Set the noncell variables to zero and determine the product $D := \prod_{i \notin \mathcal{E}} d_i$
of the minimal powers of the noncell variables.

2.2 Compute a lexicographic Gröbner basis and
solve the lattice ideal of the cellular component, adjoining roots of unity.

2.3. Save each solution D times.

3. Compute the least common multiple m of the powers of the adjoined roots of unity
and construct the cyclotomic field $\mathbb{Q}(w_m)$.
4. Return the list of solutions as elements in $\mathbb{Q}(w_m)$.

4 Using Macaulay 2

The package `Binomials` of Thomas Kahle [4, 5] is available in Macaulay 2, since version 1.4. We run some examples of [4] below.

```
i1 : S = QQ[x,y,z];
i2 : I = ideal(x^2-y,y^3-z,x*y-z);
i3 : loadPackage "Binomials";
i4 : binomialSolve I
BinomialSolve created a cyclotomic field of order 3

o4 = {{1, 1, 1}, {- ww  - 1, ww , 1},
                      3        3
    {ww , - ww  - 1, 1},
       3      3
    {0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
i5 : degree I
o5 = 6
```

We continue the session computing a binomial primary decomposition:

```
i6 : BPD I
Running cellular decomposition:
cellular components found: 1
cellular components found: 2
Decomposing cellular components:
Decomposing cellular component: 1 of 2
1 monomial to consider for this cellular component
BinomialSolve created a cyclotomic field of order 3
done
Decomposing cellular component: 2 of 2
3 monomials to consider for this cellular component
done
Removing redundant components...
4 Ideals to check
3 Ideals to check
2 Ideals to check
1 Ideals to check
0 redundant ideals removed. Computing mingens of result.

o6 = {ideal (z - 1, y - 1, x - 1),

   ideal (z - 1, y - ww , x + ww  + 1),
                       3        3

   ideal(z - 1, y + ww  + 1, x - ww ),
                      3            3

              2        2
   ideal (z, y , x*y, x  - y)}
```

We consider the last ideal in the primary decomposition

```
i7 : I = ideal(z,y^2,x*y,x^2 - y);
i8 : binomialAssociatedPrimes I
3 monomials to consider for this cellular component

o8 = {ideal (z, y, x)}
```

Cellular decompositions are computed as follows:

```
i1 : loadPackage "Binomials";
i2 : S = QQ[x1,x2,x3,x4,x5];
i3 : I = ideal(x1*x4^2-x2*x5^2, x1^3*x3^3-x2^4*x4^2, x2*x4^8-x3^3*x5^6);
i4 : I
                  2        2    3  3     4  2       8     3  6
o4 = ideal (x1*x4  - x2*x5 , x1 x3  - x2 x4 , x2*x4  - x3 x5 )
i5 : BCD I
cellular components found: 1
redundant component
redundant component
cellular components found: 2

                   2        2    3  3     4  2    3  4     2  3  2    2  6
o5 = {ideal (x1*x4  - x2*x5 , x1 x3  - x2 x4 , x2 x4  - x1 x3 x5 , x2 x4  -
     
          3  4       8     3  6
     x1*x3 x5 , x2*x4  - x3 x5 ),

                2       2        2    5    6    4  2    8
     ideal (x1 , x1*x4  - x2*x5 , x2 , x5 , x2 x4 , x4 )}
```

5 Assignments

1. Explore the package Binomials in Macaulay2.
2. Explore the capabilities in CoCoA for handling binomial ideals.
3. Explore the capabilities in Sage/Singular for binomial ideals.

References

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