

This is a take home exam. Your answers to the questions below are due on Wednesday 5 March, at 2PM (bring your answers to class). Every question counts for 20 points.

1. Apply resultants to the intersection problem of a general line  $\lambda x + \mu y = 1$  with a general circle  $(x - a)^2 + (y - b)^2 = r^2$ .
  - (a) For general values for the parameters  $\lambda$ ,  $\mu$ ,  $a$ ,  $b$ , and  $r$  derive expressions for the coordinates of the points where the line meets the circle.
  - (b) Find conditions on the parameters so the line meets the circle in a double point.
2. Reduce  $f = x_1^3 x_2 + x_1 x_2^2$  with respect to the ideal  $I(V)$  of polynomials vanishing at  $V = \{(0, 1), (1, 0), (2, 1), (-1, 2)\}$ , using a pure lexicographic order on  $x_1 < x_2$ .
  - (a) Apply interpolation to compute the reduction of  $f$ .
  - (b) Use the division algorithm on  $f$  with the Shape Lemma representation for  $I(V)$ .
3. Consider the system

$$f(x, y, \lambda) = \begin{cases} 4x - \lambda x(1 + (1/2)x) - 4 = 0 \\ 4y - \lambda y(1 + (2/3)x) - 4 = 0 \end{cases}$$

at the solution  $(x, y) = (2, 2.4)$  for  $\lambda = 1$ .

- (a) Compute the tangent vector of the path  $(x(\lambda), y(\lambda), \lambda)$  at  $(2.0, 2.4, 1.0)$ . Fix the orientation of the tangent so it has a positive component for  $\lambda$ .
  - (b) For step size 0.1, set up the equations to perform one step with Newton's method, correcting perpendicularly to the tangent direction.
4. Let  $f = x^5 y + 4x^4 y^2 - 5x^3 y^3 + 2x^3 y - 2x^2 y^2 \in \mathbb{C}[x][y]$ .
    - (a) Draw the Newton polyhedron of  $f$  and compute the leading terms of the Puiseux series expansions for  $y$  in  $x$ .
    - (b) Select a leading term and compute the second term in the expansion. Interpret the situation where  $f$  vanishes enterily at a leading term.
  5. Consider the system

$$f(x_1, x_2) = \begin{cases} \alpha_{1,1} x_1 x_2 + \alpha_{1,0} x_1 + \alpha_{0,0} = 0 \\ \beta_{1,2} x_1 x_2^2 + \beta_{1,1} x_1 x_2 + \beta_{0,0} = 0 \end{cases}$$

for nonzero choices of the coefficients  $\alpha_{1,1}$ ,  $\alpha_{1,0}$ ,  $\alpha_{0,0}$ ,  $\beta_{1,2}$ ,  $\beta_{1,1}$ , and  $\beta_{0,0}$ .

- (a) Compute the mixed volume of the Newton polygons and relate the mixed volume to the number of isolated solutions of  $f(x_1, x_2) = \mathbf{0}$ .
- (b) Find a nonzero choice of the coefficients so  $f(x_1, x_2) = \mathbf{0}$  has solutions at infinity. For this choice of coefficients, give proper homogeneous coordinates and use this homogenization to write coordinates of a solution at infinity.