Multiplicity Structure

1. Differentials and Duality
   - differential operators to define multiplicity
2. Quadruple Cyclic 9-roots
   - a system with quadruple roots
3. the Multiplicity Structure
   - computing the dimension of the dual space
   - an analysis of the deflation bound
4. A Commutative Diagram
   - relating ideal, quotient ring and dual space

MCS 563 Lecture 25
Analytic Symbolic Computation
Jan Verschelde, 12 March 2014

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defining multiplicity

A polynomial p in one variable x has an m-fold zero at z if

1. $p(x) = (x - z)^m q(x)$ and $q(z) \neq 0$; ***or***
2. $p(z) = 0$ and $\dfrac{\partial^k p}{\partial x^k}(z) = 0$, for $k = 1, 2, \ldots, m - 1$.

In both cases, we say that m is the multiplicity of the zero z.

For polynomial systems, standard bases generalize the first definition of an m-fold root.

In this lecture we will generalize the second definition with differential operators.

differential operators

For a zero $\mathbf{z} \in \mathbb{C}^n$ and a natural vector $\mathbf{a} \in \mathbb{N}^n$,
the differential operator $\partial_\mathbf{a}[\mathbf{z}]$ is defined by

$$\partial_\mathbf{a}[\mathbf{z}] : \mathbb{C}[\mathbf{x}] \rightarrow \mathbb{C} : p \mapsto (\partial_\mathbf{a} p)(\mathbf{z}),$$

with

$$\partial_\mathbf{a} p = \frac{1}{a_1! a_2! \cdots a_n!} \frac{\partial^{a_1+a_2+\cdots+a_n} p}{\partial x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}}.$$

We can write Taylor expansions in a very compact way.

dual space

Observe the linearity of the differential operator:

$$\partial_{\mathbf{a}}[\mathbf{z}](\lambda p + \mu q) = \lambda \partial_{\mathbf{a}}[\mathbf{z}](p) + \mu \partial_{\mathbf{a}}[\mathbf{z}](q).$$

A general differential operator is a linear combination of several $\partial_{\mathbf{a}}[\mathbf{z}]$'s.
The dual space $D_{\mathbf{z}}[I]$ of I at $\mathbf{z}$ is

$$D_{\mathbf{z}}[I] = \left\{ d[\mathbf{z}] = \sum_{\mathbf{a}} c_{\mathbf{a}} \partial_{\mathbf{a}}[\mathbf{z}], c_{\mathbf{a}} \in \mathbb{C} \mid d[\mathbf{z}](p) = 0, \forall p \in I \right\}.$$

It is the space of all differential operators which make every polynomial in the ideal I vanish at the root $\mathbf{z}$.

an example

Consider

$$I = \langle x_1^2, x_1x_2, x_2^2 \rangle, \quad \mathbf{z} = (0,0).$$

The dual space of I at $\mathbf{z}$ is

$$D_{\mathbf{z}}[I] = \mathrm{span}\{\ \partial_{00}[\mathbf{z}], \partial_{10}[\mathbf{z}], \partial_{01}[\mathbf{z}]\ \}.$$

Since $\dim(D_{\mathbf{z}}[I]) = 3$, the multiplicity of $\mathbf{z}$ equals three.

$p \in I$: $p = q_{20}x_1^2 + q_{11}x_1x_2 + q_{02}x_2^2$, $q_{ij} \in \mathbb{C}[\mathbf{x}]$, $\partial_{00}(p) = p$.

$$\partial_{10}(p) = x_1 \left(\frac{\partial q_{20}}{\partial x_1} x_1 + q_{20} 2 \right) + \left(\frac{\partial q_{11}}{\partial x_1} x_1 x_2 + q_{11} x_2 \right) + \frac{\partial q_{02}}{\partial x_1} x_2^2$$

Observe: $\partial_{10}[(x_1 = 0, x_2 = 0)](p) = 0$. By symmetry, $x_1 \leftrightarrow x_2$: $\partial_{01}[\mathbf{z}](p) = 0$. Higher derivatives have nonzero constants and do not vanish at $\mathbf{z}$.

a more interesting ideal

To illustrate the correspondence with standard bases:

$$I = \langle x_1^2 + 2x_2^2 - 2x_2, x_1x_2^2 - x_1x_2, x_2^3 - 2x_2^2 + x_2 \rangle$$

has two roots $\mathbf{z}_0 = (0,0)$ and $\mathbf{z}_1 = (0,1)$ with respective multiplicities $m_0 = 2$ and $m_1 = 3$.

The dual space of I at $\mathbf{z}_0$ is

$$D_{\mathbf{z}_0}[I] = \mathrm{span}\{\partial_{00}[\mathbf{z}_0], \partial_{10}[\mathbf{z}_0]\}$$

and the dual space of I at $\mathbf{z}_1$ is

$$D_{\mathbf{z}_1}[I] = \mathrm{span}\{\partial_{00}[\mathbf{z}_1], \partial_{10}[\mathbf{z}_1], 2\partial_{20}[\mathbf{z}_1] - \partial_{01}[\mathbf{z}_1]\}.$$

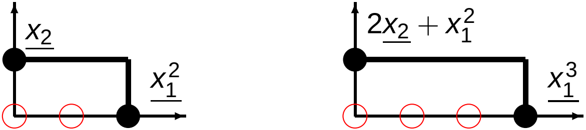
The dual space of I is $D[I] = D_{\mathbf{z}_0}[I] \cup D_{\mathbf{z}_1}[I]$.

relation to standard bases

$D[I] = D_{\mathbf{z}_0}[I] \cup D_{\mathbf{z}_1}[I]$ with $D_{\mathbf{z}_0}[I] = \mathrm{span}\{\partial_{00}[\mathbf{z}_0], \partial_{10}[\mathbf{z}_0]\}$

and $D_{\mathbf{z}_1}[I] = \mathrm{span}\{\partial_{00}[\mathbf{z}_1], \partial_{10}[\mathbf{z}_1], 2\partial_{20}[\mathbf{z}_1] - \partial_{01}[\mathbf{z}_1]\}$.

Standard bases for $\mathbf{z}_0$ and $\mathbf{z}_1$ (after shift) for I:



For every leading term $\partial_{\mathbf{a}}$ in a generator of $D_{\mathbf{z}}[I]$,
there is a corresponding standard monomial $\mathbf{x}^{\mathbf{a}}$,
where $\mathbf{a}$ is marked by a dark circle.

Multiplicity Structure

cyclic 9-roots

A system of 9 equations in 9 unknowns:

$$f(\mathbf{z}) = \left\{ \begin{array}{r} z_0 + z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + z_7 + z_8 = 0 \\ z_0 z_1 + z_1 z_2 + z_2 z_3 + z_3 z_4 + z_4 z_5 + z_5 z_6 \\ + z_6 z_7 + z_7 z_8 + z_8 z_0 = 0 \\ \text{for } i = 3, 4, \ldots, 8 : \displaystyle\sum_{j=0}^{8} \prod_{k=j}^{i} z_{k \bmod 9} = 0 \\ z_0 z_1 z_2 z_3 z_4 z_5 z_6 z_7 z_8 - 1 = 0 \end{array} \right.$$

There are 162 roots of multiplicity four.
One deflation locates every 4-fold root accurately.
Running the algorithm of Dayton and Zeng yields

$$H[1] = 1, H[2] = 2, H[3] = 1, H[4] = 0,$$
$$\text{with } H[i] = \text{Nullity}(S_i) - \text{Nullity}(S_{i-1}), i > 0.$$

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the dimension of $D_{\mathbf{z}}[I]$

We look for differentiation functionals $d[\mathbf{z}] = \sum_{\mathbf{a}} c_{\mathbf{a}} \partial_{\mathbf{a}}[\mathbf{z}]$.

For $I = \langle f_i, i = 1, 2, \ldots, N \rangle$, the membership criterion is

$$d[\mathbf{z}] \in D_{\mathbf{z}}[I] \Leftrightarrow d[\mathbf{z}](pf_i) = 0, \forall p \in \mathbb{C}[\mathbf{x}], i = 1, 2, \ldots, N.$$

To turn this criterion into an algorithm, observe:

1. $d[\mathbf{z}]$ is linear, we restrict p to $\mathbf{x}^{\mathbf{k}} = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$; and
2. we limit degrees $k_1 + k_2 + \cdots + k_n \leq a_1 + a_2 + \cdots + a_n$, as $\mathbf{z} = \mathbf{0}$ vanishes trivially if not annihilated by $\partial_{\mathbf{a}}$.

By these turns, we consider the application of a differential operator to monomial multiples of the polynomials generating the ideal, evaluated at the root $\mathbf{z}$.

Multiplicity Matrices

We define a sequence of *multiplicity matrices* S_k,

- where the columns of S_k are indexed by $\partial_{\mathbf{a}}$, for $|\mathbf{a}| \leq k$,
- and the rows by $\mathbf{x}^{\mathbf{b}} f_i$, for $|\mathbf{b}| \leq k - 1$.

The entries of S_k are the values of $\partial_{\mathbf{a}}(\mathbf{x}^{\mathbf{b}} f_i)$, evaluated at $\mathbf{z}$.

By convention, $S_0 = f(\mathbf{z})$. The null space of the S_k's yield generators for the dual space $D_{\mathbf{z}}[I]$.

Since the dual space is finitely generated, the algorithm stops when the nullity (dimension of the null space) does not increase between S_{k-1} and S_k.

Theorem (the Hilbert function)

The Hilbert function $H(k)$ and multiplicity m are

$$H(k) = \text{nullity}(S_k) - \text{nullity}(S_{k-1}), k = 1, 2, \ldots \quad m = \sum_{k=1}^{\infty} H(k).$$

an example

Let $f_1 = x_1 - x_2 + x_1^2$, $f_2 = x_1 - x_2 + x_2^2$, $\mathbf{z} = (0,0)$.

		$\|a\|=0$	$\|a\|=1$		$\|a\|=2$			$\|a\|=3$			
		∂_{00}	∂_{10}	∂_{01}	∂_{20}	∂_{11}	∂_{02}	∂_{30}	∂_{21}	∂_{12}	∂_{03}
	f_1	0	1	−1	1	0	0	0	0	0	0
S_1	f_2	0	1	−1	0	0	1	0	0	0	0
	$x_1 f_1$	0	0	0	1	−1	0	1	0	0	0
	$x_1 f_2$	0	0	0	1	−1	0	0	0	1	0
	$x_2 f_1$	0	0	0	0	1	−1	0	1	0	0
S_2	$x_2 f_2$	0	0	0	0	1	−1	0	0	0	1
	$x_1^2 f_1$	0	0	0	0	0	0	1	−1	0	0
	$x_1^2 f_2$	0	0	0	0	0	0	1	−1	0	0
	$x_1 x_2 f_1$	0	0	0	0	0	0	0	1	−1	0
	$x_1 x_2 f_2$	0	0	0	0	0	0	0	1	−1	0
	$x_2^2 f_1$	0	0	0	0	0	0	0	0	1	−1
S_3	$x_2^2 f_2$	0	0	0	0	0	0	0	0	1	−1

the example continued

$$f(\mathbf{x}) = \begin{cases} f_1 = x_1 - x_2 + x_1^2 = 0 \\ f_2 = x_1 - x_2 + x_2^2 = 0 \end{cases} \quad \mathbf{z} = (0,0).$$

As Nullity(S_2) = Nullity(S_3), the algorithm stops.

The vectors in the kernel of the multiplicity matrices lead to the generators of the dual space

$$D_{\mathbf{z}}[I] = \mathrm{span}\{\ \partial_{00}, \partial_{10} + \partial_{01}, -\partial_{10} + \partial_{20} + \partial_{11} + \partial_{02}\ \}.$$

As there are 3 generators, the multiplicity equals 3.

The nullity (or the corank) of a matrix is computed via rank-revealing algorithms using the QR decomposition.

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breadth and depth

The number of stages in the deflation method to recondition an isolated singular solution is bounded by the multiplicity.

Dayton & Zeng introduce the *breadth* and *depth* of $D_{\mathbf{z}}[I]$ as

$$\mathrm{breadth}_{\mathbf{z}}[I] = H(1) \quad \text{and} \quad \mathrm{depth}_{\mathbf{z}}[I] = \max\{\, \alpha \mid H(\alpha) > 0 \,\},$$

where $H(\cdot)$ is the Hilbert function.

Theorem (Dayton and Zeng, 2005)

The number of deflation stages for an isolated solution $\mathbf{z} \in V(I)$ is bounded by $\mathrm{depth}_{\mathbf{z}}(I)$.
Furthermore, if k stages are executed in the deflation method, then 2^k differential functionals in $D_{\mathbf{z}}[I]$ have been computed by the deflation method.

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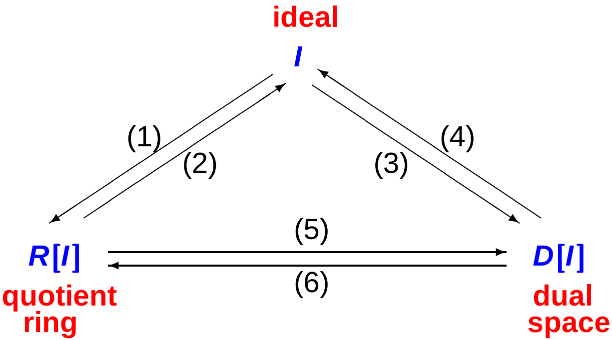
a commutative diagram

A commutative diagram between ideal I,
the residue class ring $R[I]$ and the dual space $D[I]$:



going back and forth

Relations between I and $R[I]$:

(1) $\boldsymbol{I \longrightarrow R[I]}$: the residue classes of Lagrange polynomials interpolating at $V(I)$ give a basis for $R[I]$.
$R[I] = \{ [p]_I \mid p \in \mathbb{C}[\mathbf{x}] \}$, where the residue class of p is $[p]_I = \{ r \in \mathbb{C}[\mathbf{x}] \mid p - r \in I \}$.

(2) $\boldsymbol{R[I] \longrightarrow I}$: for some basis $\mathbf{b}$ of $R[I]$: $A_k \mathbf{b} = x_k \mathbf{b}$, $k = 1, 2, \ldots, n$, means $x_k \mathbf{b} = A_k \mathbf{b}$ mod I, or $x_k \mathbf{b} - A_k \mathbf{b} = 0$ over $V(I)$, and $x_k \mathbf{b} - A_k \mathbf{b} \in I$ leads to a border basis for I.

going back and forth

Relations between I and $D[I]$:

(3) $\boldsymbol{I \longrightarrow D[I]}$: evaluating at the zeroes $\mathbf{z} \in V(I)$ gives a natural basis for $D[I]$, m-fold zeroes are represented by m differential operators.

(4) $\boldsymbol{D[I] \longrightarrow I}$: we have the following

$$\begin{aligned} I[D[I]] &= \ker(D[I]) \\ &= \{\, p \in \mathbb{C}[\mathbf{x}] : \ell(p) = 0, \forall \ell \in D[I] \,\} \end{aligned}$$

going back and forth

Relations between $R[I]$ and $D[I]$:

(5) ***R[I]* $\longrightarrow$ *D[I]***: $(R[I])^* = D[I]$. The monomials in a normal set (basis for the residue class ring $R[I]$) correspond to leading terms of differential operators.

(6) ***D[I]* $\longrightarrow$ *R[I]***: $(D[I])^* = R[I]$. The leading terms of the differential operators that generate the dual space $D[I]$ correspond to basis elements for the quotient ring $R[I]$.

Summary + Exercises

The dimension of the dual space at a root determines its multiplicity. Generators of the dual space are found in the kernel of multiplicity matrices.

Exercises:

1. For a polynomial p in one variable with an m-fold zero at z prove $p(x) = (x - z)^m q(x)$ and $q(z) \neq 0$

$$\Leftrightarrow \quad p(z) = 0 \text{ and } \frac{\partial^k p}{\partial x^k}(z) = 0, \text{ for } k = 1, 2, \ldots, m - 1,$$

   i.e.: these are equivalent definitions of multiplicity m.
2. Take $I = \langle x_1^3 + x_1 x_2^2, x_1 x_2^2 + x_2^3, x_1^2 x_2 + x_1 x_2^2 \rangle$.
   Describe the dual space $D_{\mathbf{0}}[I]$.

more exercises

3. Compare the multiplicity structure of the ideals $I_1 = \langle x_1^3, x_2^4 \rangle$ and $I_2 = \langle x_1^3, x_2^4 + x_1^2 x_2 \rangle$.
    1. What is the multiplicity of $(0,0)$ for I_1 and I_2?
    2. Compute $D_{\mathbf{0}}[I_1]$ and $D_{\mathbf{0}}[I_2]$.
4. For the computations of the previous exercise, make a cost analysis of the rank computations, taking into account the steady increase of dimensions of the matrices. Estimate the cost benefit of working incrementally, updating QR decompositions with new rows and columns, instead of recomputing each time from scratch.

one last exercise

5. Consider the system $f(x,y) = \begin{cases} x^2 + y - 3 = 0 \\ x + 0.125y^2 - 1.5 = 0. \end{cases}$

   This system has one regular solution and a triple root at $(1,2)$.

   1. Set up the multiplicity matrix at $(1,2)$ and use it to compute the multiplicity of this root.
   2. Instead of $(1,2)$, evaluate the multiplicity matrix at $(0.9999999, 2.000000001)$. How large should the tolerance ϵ (used to decide the numerical rank) be in order to obtain correct results?