

# Polyhedral Homotopies

- 1 Bernshtein's first theorem
  - mixed volumes bound #isolated solutions
  - polyhedral homotopies lead to all isolated solutions
- 2 Runge-Kutta Formulas
  - solving initial value problems
- 3 Regular Mixed-Cell Configurations
  - computing mixed cells and mixed volumes
  - patchworking algebraic curves

MCS 563 Lecture 19  
Analytic Symbolic Computation  
Jan Verschelde, 26 February 2014

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# Bernshtein's first theorem

Let  $f(\mathbf{x}) = \mathbf{0}$  be a system of  $n$  equations  $f = (f_1, f_2, \dots, f_n)$  in  $n$  unknowns  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  and complex coefficients.

The supports of  $f$  are in  $\mathcal{A} = (A_1, A_2, \dots, A_n)$ :

$$f_i(\mathbf{x}) = \sum_{\mathbf{a} \in A_i} c_{i\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad c_{i\mathbf{a}} \in \mathbb{C}^*, \quad \mathbb{C}^* = \mathbb{C} \setminus \{0\}, \quad i = 1, 2, \dots, n.$$

The supports  $\mathcal{A}$  span the Newton polytopes  $\mathcal{P} = (P_1, P_2, \dots, P_n)$ ,  $P_i = \text{conv}(A_i)$ ,  $i = 1, 2, \dots, n$ .

We denote the mixed volume of  $\mathcal{P}$  by  $V(\mathcal{P})$ .

## Theorem (Bernshtein's theorem A)

*#isolated solutions in  $(\mathbb{C}^*)^n$  of  $f(\mathbf{x}) = \mathbf{0} \leq V(\mathcal{P})$ .*

A constructive proof uses polyhedral homotopies.

## the cheater homotopy

Let  $g(\mathbf{x}) = \mathbf{0}$  be a system with the same Newton polytopes as  $f(\mathbf{x}) = \mathbf{0}$  but with random complex coefficients.

$$h(\mathbf{x}, t) = (1 - t)g(\mathbf{x}) + t f(\mathbf{x}) = 0, \quad t \in [0, 1].$$

If  $g(\mathbf{x}) = \mathbf{0}$  has exactly as many solutions in  $(\mathbb{C}^*)^n$  as  $V(\mathcal{P})$ , then Bernshtein's first theorem is implied by the coefficient-parameter homotopy  $h(\mathbf{x}, t) = \mathbf{0}$ .

We apply Bernshtein's second theorem to  $g(\mathbf{x}) = \mathbf{0}$ .

For  $\mathbf{v} \in \mathbb{Z}^n$ ,  $\mathbf{v} \neq \mathbf{0}$ , consider initial forms in  $\text{in}_{\mathbf{v}}g(\mathbf{x}) = \mathbf{0}$ .

Since all monomials  $\mathbf{x}^{\mathbf{a}}$  of  $\text{in}_{\mathbf{v}}g_i$  yield the same  $\langle \mathbf{a}, \mathbf{v} \rangle$ , for  $\mathbf{a} \in A_i$ , we define a unimodular coordinate transformation to eliminate one variable.

An overdetermined system of  $n$  equations in  $n - 1$  variables with random coefficients has no solutions.

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# polyhedral homotopies

We first solve  $g(\mathbf{x}) = \mathbf{0}$  via polyhedral homotopies.

To the supports of  $g$  we apply a lifting function  $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ ,  
 $\omega_j : A_j \rightarrow \mathbb{Z} : \mathbf{a} \mapsto \omega_j(\mathbf{a})$ .

This leads to the system  $\hat{g}(\mathbf{x}, t)$  with equations

$$\hat{g}_i(\mathbf{x}, t) = \sum_{\mathbf{a} \in A_i} \bar{c}_{i\mathbf{a}} \mathbf{x}^{\mathbf{a}} t^{\omega_i(\mathbf{a})}, \quad \bar{c}_{i\mathbf{a}} \in \mathbb{C}^*,$$

where the coefficients  $\bar{c}_{i\mathbf{a}}$  are random complex numbers.

To solve  $\hat{g}(\mathbf{x}, t) = \mathbf{0}$ , we look for inner normals  $\mathbf{v} = (\mathbf{u}, 1)$  for which the corresponding initial form system  $\text{in}_{\mathbf{v}} \hat{g}(\mathbf{x}, t) = \mathbf{0}$  has a solution in  $(\mathbb{C}^*)^n$ .

## changing coordinates

Changing coordinates  $x_j = y_j t^{v_j}$ ,  $j = 1, 2, \dots, n$ , then

$$\begin{aligned}\widehat{g}_i(\mathbf{y}, t) &= \sum_{\mathbf{a} \in A_i} \bar{c}_{i\mathbf{a}} (y_1 t^{v_1} y_2 t^{v_2} \dots y_n t^{v_n})^{\mathbf{a}} t^{\omega_i(\mathbf{a})} \\ &= \sum_{\mathbf{a} \in A_i} \bar{c}_{i\mathbf{a}} \mathbf{y}^{\mathbf{a}} t^{v_1 a_1 + v_2 a_2 + \dots + v_n a_n + \omega_i(\mathbf{a})} \\ &= \sum_{\mathbf{a} \in A_i} \bar{c}_{i\mathbf{a}} \mathbf{y}^{\mathbf{a}} t^{\langle \mathbf{a}, \mathbf{v} \rangle + \omega_i(\mathbf{a})}.\end{aligned}$$

As  $\mathbf{v}$  determines the coordinates change, denote

$\widehat{g}_{\mathbf{v}}(\mathbf{y}, t) = \widehat{g}_i(x_j = y_j t^{v_j})$ , and  $m_i = \min_{\mathbf{a} \in A_i} \langle \mathbf{a}, \mathbf{v} \rangle$ , so: monomials of  $\widehat{g}_{\mathbf{v}}$  with

lowest exponent  $m_i$  belong to  $\text{in}_{\mathbf{v}} \widehat{g}_{\mathbf{v}}$ .

Thus  $(t^{-m_i} \widehat{g}_{\mathbf{v},i})(\mathbf{y}, 0) = \text{in}_{\mathbf{v}}(t^{-m_i} \widehat{g}_{\mathbf{v},i})(\mathbf{y})$ .

Initial forms of  $\widehat{g}(\mathbf{x}, t)$  are start systems.

The  $\mathbf{v}$  are the leading powers of Puiseux series.

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# Runge-Kutta formulas

To solve an initial value problem  $y'(t) = f(t, y(t))$ ,  $y(t_0) = y_0$ ,  
Runge-Kutta methods approximate  $y(t)$ , at discrete times  
 $t_{k+1} = t_k + h_k$ ,  $k = 0, 1, \dots$

A Runge-Kutta formula of order  $s$  has the form

$$y_{k+1} = y_k + h_k \sum_{i=1}^s b_i f_i \approx y(t_{k+1}) = y(t_k + h_k),$$

with

$$f_i = f \left( t_k + c_i h_k, y_k + \sum_{j=1}^s a_{ij} f_j \right), \quad i = 1, 2, \dots, s.$$

We determine  $c_i$  and  $a_{ij}$  requiring that  $y_{k+1}$  for  $y(t_{k+1})$  matches the Taylor expansion of  $y(t_{k+1})$ .

## a polynomial system

For  $s = 2$ , we have the following Runge-Kutta formula:

$$\begin{aligned}y_{k+1} &= y_k + b_1 k_1 + b_2 k_2 \\k_1 &= hf(t_k, y_k) \\k_2 &= hf(t_k + \alpha h, y_k + \beta k_1),\end{aligned}$$

using a fixed step size  $h$ . We can then derive

$$b_1 + b_2 = 1, \quad \alpha b_2 = \frac{1}{2}, \quad \beta b_1 = \frac{1}{2},$$

which defines a whole family of formulas with local error  $O(h^3)$ . Any member in the family defines a second-order Runge-Kutta method.

Only solutions for which all coordinates are nonzero matter.

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## computing mixed cells

Because only in solutions in  $(\mathbb{C}^*)^n$  matter, every equation of  $\text{in}_{\mathbf{v}}\widehat{g}$  must have at least two monomials.

The exponent of those two monomials in  $\text{in}_{\mathbf{v}}\widehat{g}_i$  span an edge on the lower hull of  $\partial_{\mathbf{v}}\widehat{P}_i$ , for  $i = 1, 2, \dots, n$ .

Denote the edge which spans  $\partial_{\mathbf{v}}\widehat{P}_i$  by  $\partial_{\mathbf{v}}\widehat{A}_i = \{\widehat{\mathbf{a}}, \widehat{\mathbf{b}}\}$ .

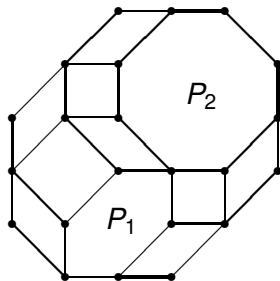
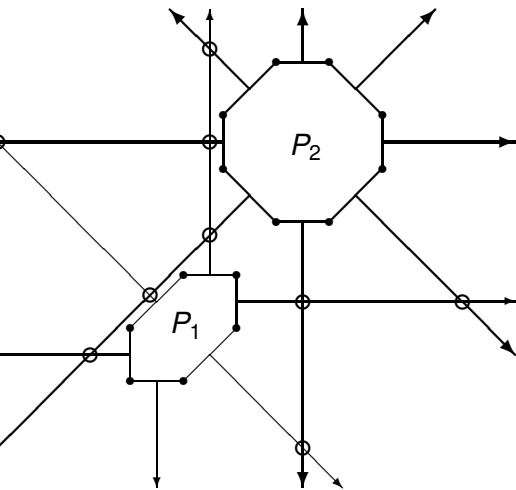
Then the inner normal  $\mathbf{v}$  to this edge satisfies

$$\begin{cases} \langle \widehat{\mathbf{a}}, \mathbf{v} \rangle = \langle \widehat{\mathbf{b}}, \mathbf{v} \rangle \\ \langle \widehat{\mathbf{a}}, \mathbf{v} \rangle \leq \langle \widehat{\mathbf{c}}, \mathbf{v} \rangle, \end{cases} \quad \text{for all } \mathbf{c} \in A_i.$$

Enumerating all edges of a polytope is thus equivalent to enumerating all feasible solutions to the linear inequalities.

Letting  $i$  range from 1 to  $n$  applied to the lifted point sets  $\widehat{A}_i$  provides the dual linear-programming model to enumerate all inner normals to the mixed cells in a regular mixed subdivision: a mixed-cell configuration.

# outer normal cones

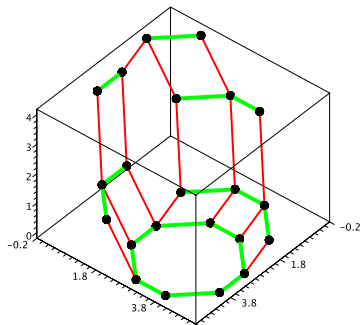
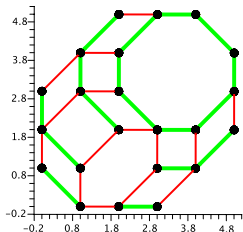


## a regular mixed subdivision

$$A_1 = \{(2, 2), (1, 2), (2, 1), (0, 1), (1, 0), (0, 0)\} \text{ and}$$

$$A_2 = \{(2, 3), (1, 3), (3, 2), (0, 2), (3, 1), (0, 1), (2, 0), (1, 0)\}$$

span  $P_1$  and  $P_2$ . We lift  $P_1$  with  $\omega : (a_1, a_2) \mapsto 4 - a_1/2 - 3a_2/2$  while  $P_2$  stays at height 0.



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## patchworking algebraic curves

Regular mixed subdivisions were originally defined to generalize the Viro method to isolated solutions.

We consider a polynomial with real coefficients. Its signed Newton polygon has at each vertex the sign (+ or -) of the corresponding coefficient.

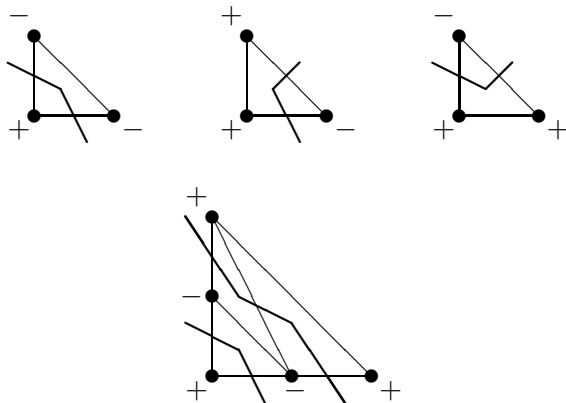
Transferring the polygon to other quadrants by flipping over the coordinate axes, copying the signs of even exponents and flipping the signs of odd exponents.

In a regular triangulation we connect midpoints of edges with vertices of opposite sign to the barycenter of the triangles that contain the edges.

The lines drawn are a piecewise-linear sketch of a curve.

A polyhedral homotopy realizes the topology of the curve.

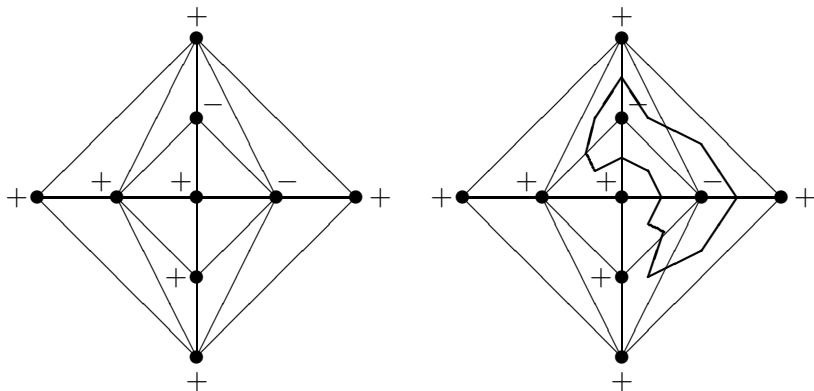
# signed Newton polygons



$$f(x, y) = +c_{2,0}tx^2 + c_{0,2}y^2 - c_{1,0}x - c_{0,1}y + c_{0,0}t, \quad c_{i,j} > 0, \quad t = \epsilon$$

# Viro diagrams

$$f(x, y) = +c_{2,0}tx^2 + c_{0,2}y^2 - c_{1,0}x - c_{0,1}y + c_{0,0}t, \quad c_{i,j} > 0, \quad t = \epsilon$$



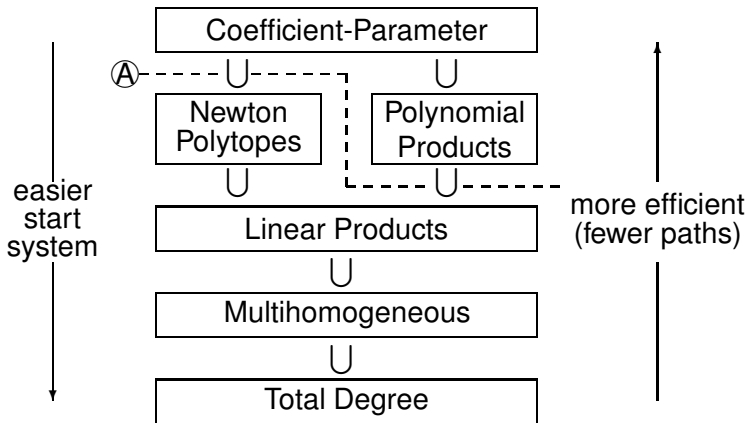
# Newton polytopes in polymake

```
polytope > $r = new Ring(qw(x y t));
polytope > ($x,$y,$t) = $r->variables;
polytope > $f = $t*$x*$x + $y*$y - $x - $y + $t;
polytope > print($f);
-1*x -1*y + t + y^2 + x^2*t
polytope > $p = newton($f);
polytope > print $p->VERTICES;
polymake: used package cddlib
  Implementation of the double description method
  of Motzkin et al. Copyright by Komei Fukuda.
  http://www.ifor.math.ethz.ch/~fukuda/cdd_home/cdd.html
```

```
1 1 0 0
1 0 1 0
1 0 0 1
1 0 2 0
1 2 0 1
```

```
polytope > $p->VISUAL;
```

# a totem pole of homotopies



# Summary + Exercises

Polyhedral homotopies provide proof that mixed volumes count the roots of random coefficient polynomial systems. Mixed-cell configurations store the supports of all start systems in polyhedral homotopies.

## Exercises:

- 1 Derive the conditions for the third and fourth order Runge-Kutta formulas, i.e.:  $s = 3$  and  $s = 4$ .
- 2 Consider the polynomial system

$$f(\mathbf{x}) = \begin{cases} x_1 x_2 + x_1 + x_2 + 1 = 0 \\ x_1^2 x_2^2 + x_1 + x_2 = 0. \end{cases}$$

Choose a lifting of the points in the supports of  $f$  and compute its mixed-cell configuration. List all homotopies and start systems used to solve a random coefficient system  $g$  for  $f$ .

## two more exercises

- 3 Give the system of the previous exercise as input to `phc -m`. Compute a mixed-cell configuration.
- 4 Consider the polygons in the picture with the outer normal cones. Select one intersection of outward pointing normal vectors and setup the corresponding system of linear inequalities. Compute the inner normal to the corresponding mixed cell of the regular mixed subdivision induced for the two polygons.