Review and Summary of the first part of MCS 563

Our object of study is a system $f(\mathbf{x}) = \mathbf{0}$ of N polynomials $f = (f_1, f_2, \ldots, f_N)$, with complex coefficients: $f_i \in \mathbb{C}[\mathbf{x}]$, $i = 1, 2, \ldots, N$, in n variables: $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. We are interested in algorithms to compute and describe the solution set $V = f^{-1}(\mathbf{0})$.

The topics are summarized in the titles of the lectures:

Lec 1	introduction (Newton and Bézout)	Lec 2	elimination methods (resultants)
Lec 3	homotopies and predictor-corrector	Lec 4	rewriting polynomials (division algorithm)
Lec 5	alpha theory to certify roots	Lec 6	Gröbner bases (Buchberger's algorithm)
Lec 7	multihomogenization (linear-product)	Lec 8	quotient rings (shape lemma)
Lec 9	Condition and Scaling	Lec 10	Gröbner basis conversion (FGLM)
Lec 11	coefficient-parameter or cheater	Lec 12	Rational Univariate Representation
Lec 13	Real Homotopies (turning points)	Lec 14	Kronecker Parametrization (Newton-Hensel)
Lec 15	the Newton-Puiseux method		
Lec 16	Kushnirenko's theorem (Newton polytopes)		
Lec 17	mixed volumes (Cayley trick)		
Lec 18	Bernshteǐn's second theorem		
Lec 19	Polyhedral Homotopies		

We can classify the topics in three categories:

Numerical Homotopy Continuation: Starting in the first lecture with Newton's method, homotopies were introduced in lecture three. Every odd numbered lecture from 3 till 13 considered polynomial systems from a numerical perspective.

Symbolic Rewriting and Elimination: In lecture 2 we used resultants to introduce the main theorem of elimination theory. In every even numbered lecture from 2 till 14, we looked at polynomial systems from a symbolic perspective, using term rewriting techniques as solution methods.

Polyhedral Methods: Newton polytogons arose in lecture 15 in a method to compute Puiseux series. In the last four lectures, we stated the theorems of Kushnirenko and Bernshteǐn, ending with polyhedral homotopies.

Continuing the classification of materials in a tripartite fashion, we distinguish concepts, theorems, and algorithms. The exercises usually provide more examples of definitions encountered in the lectures and give opportunities to explore the algorithms.

A typical question we ask when we see a polynomial system is to bound the number of isolated solutions. We have seen two generalizations of Bézout's theorem and used mixed volumes to compute a root count. With every root count we associate a start system to be used in a homotopy.

An alternative look at a polynomial system applies elimination, either explicitly via resultants or the shape lemma, or more implicitly via eigenvalue problems and rational univariate parameterization. The main tool here is a Gröbner basis which turns the division algorithm into a normal form algorithm allowing computations in the quotient ring.