

# Standard Bases

- 1 Localization and Multiplicities
  - investigating local properties
- 2 Singular Points on Curves and Surfaces
  - visualization of algebraic surfaces
- 3 Mora's Normal Form Algorithm
  - a weak normal form
- 4 Computing Multiplicities
  - multiplicity as the dimension of the local quotient ring

MCS 563 Lecture 23  
Analytic Symbolic Computation  
Jan Verschelde, 7 March 2013

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## 1 Localization and Multiplicities

- investigating local properties

## 2 Singular Points on Curves and Surfaces

- visualization of algebraic surfaces

## 3 Mora's Normal Form Algorithm

- a weak normal form

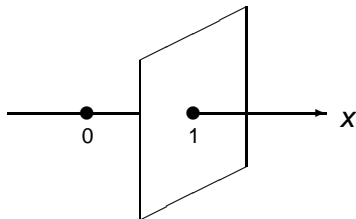
## 4 Computing Multiplicities

- multiplicity as the dimension of the local quotient ring

## an example

Consider  $I = \langle y(x - 1), z(x - 1) \rangle \subset \mathbb{Q}[x, y, z]$ .

The real picture of  $V(I)$  or  $\begin{cases} y(x - 1) = 0 \\ z(x - 1) = 0 \end{cases}$  is below:



The local dimension of  $V(I)$  at  $(0, 0, 0)$  is 1  
and is 2 at  $(1, 0, 0)$ .

# localization

The localization of  $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, x_2, \dots, x_n]$   
at  $\langle \mathbf{x} \rangle = \langle x_1, x_2, \dots, x_n \rangle$  is

$$\mathbb{C}[\mathbf{x}]_{\langle \mathbf{x} \rangle} = \left\{ \frac{f(\mathbf{x})}{g(\mathbf{x})} \mid f, g \in \mathbb{C}[\mathbf{x}], g(\mathbf{0}) \neq 0 \right\}.$$

Instead of at  $\langle \mathbf{x} \rangle$  we can localize from any point  $\mathbf{z} \in \mathbb{C}^n$ , not just the origin  $\mathbf{0}$ , shifting the coordinate system  $x_i = y_i - z_i$ ,  $i = 1, 2, \dots, n$ .

Using local monomial orderings (Singular: `ds`, `Ds`, `ls`)  
we compute in  $\mathbb{C}[\mathbf{x}]_{\langle \mathbf{x} \rangle}$  without denominators.

## local monomial orders

Consider the ideal  $I = \langle x_1^2 + x_1^3, x_2^2 \rangle$ .

Factoring the first polynomial as  $x_1^2(1 + x_1)$ , we see that  $V(I)$  consists of two distinct roots.

To focus on the singular solution at the origin, we look at the lowest powers of the monomials instead of the highest ones.

Therefore we order the terms in the opposite order, equivalent of taking negative weights.

One global term order is lexicographic, denoted by  $>_{\text{lex}}$ .

The corresponding local term order  $\text{neglex}$  is  $\mathbf{x}^{\mathbf{a}} >_{\text{neglex}} \mathbf{x}^{\mathbf{b}}$  if the leftmost nonzero entry in  $\mathbf{a} - \mathbf{b}$  is negative.

For example:  $x_1 x_2^3 >_{\text{neglex}} x_1^2 x_2$ .

## computing multiplicities

For  $I = \langle x_1^2 + x_1^3, x_2^2 \rangle$  a Gröbner basis leads to the initial ideal  $\langle x_1^3, x_2^2 \rangle$  which shows there are six solutions.

Using a local term order, a local analogue to a Gröbner basis is a standard basis.

A standard basis for  $I$  gives the monomial ideal  $\langle x_1^2, x_2^2 \rangle$ .

If  $\#V(I) < \infty$ , a Gröbner basis gives the dimension of the quotient ring  $\mathbb{C}[\mathbf{x}]/I$  and  $\#V(I)$ .

Monomials not in  $\langle LT(I) \rangle$  define a basis for  $\mathbb{C}[\mathbf{x}]/I$ .

Analogously to the computation of the dimension of the quotient ring via a Gröbner basis, with a standard basis we compute the multiplicity of a point.

For  $I = \langle x_1^2 + x_1^3, x_2^2 \rangle$ ,  $(0, 0) \in V(I)$  has multiplicity 4.

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## singularities on surfaces

A point  $\mathbf{z} \in \mathbb{C}^n$  on a hypersurface  $f(\mathbf{x}) = 0$  is singular

- if  $f(\mathbf{z}) = 0$  and
- also all its partial derivatives vanish:  $\frac{\partial f}{\partial x_i}(\mathbf{z}) = 0, i = 1, 2, \dots, n.$

The singularity is isolated if there exists a ball around the singular point that contains no other singular point.

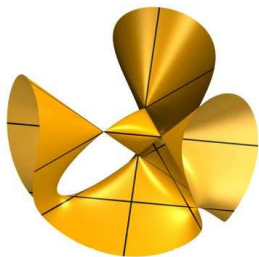
The affine equation for the four-nodal Cayley cubic is

$$4 \left( x^3 + 3x^2 - 3xy^2 + 3y^2 + \frac{1}{2} \right) + 3 \left( x^2 + y^2 \right) (z - 6) - z \left( 3 + 4z + 7z^2 \right) = 0.$$

This surface contains exactly three lines of multiplicity one and six lines of multiplicity four.

## visualizations

From the thesis of Oliver Labs ([www.OliverLabs.net](http://www.OliverLabs.net)):

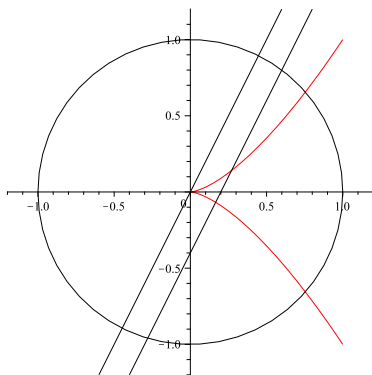


On the left is the four-nodal Cayley cubic and its 9 lines.

The other picture is a 16-nodal Kummer surface, known to Kummer in 1864, a quartic surface in 3-space with the maximum number of nodes.

## a cuspidal cubic

$$f = x^3 - y^2 = 0 \cap x + \frac{1}{2}y = 0 \text{ and } \cap x + \frac{1}{2}y = \epsilon:$$



Restricting  $f$  to the shifted line

$$f(x = \epsilon - \frac{1}{2}y, y) = (\epsilon - \frac{1}{2}y)^3 - y^2 = -y^2(\frac{1}{8}y + 1) + \frac{3}{4}y^2\epsilon + O(\epsilon^2)$$

and by the two zeroes close to  $(0, 0)$ : multiplicity 2.

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## a weak normal form

Instead of  $x = \left( \sum_{k=0}^{\infty} x^k \right) (x - x^2)$

we write  $(1 - x)x = x - x^2$ .

This gives rise to a weak normal form:

$$uf = \sum_{i=1}^s a_i g_i + \text{NF}(f|G), \quad G = \{g_1, g_2, \dots, g_s\}.$$

We define  $\text{ecart}(f) := \deg(f) - \deg(\text{LM}(f))$ .

For a homogeneous polynomial  $f$  we have  $\text{ecart}(f) = 0$ .

Mora's normal form algorithm returns a weak normal form of a polynomial with respect to a set of polynomials.

This algorithm is analogous to the division algorithm.

# Mora's NF algorithm

NFMora( $f, G, >$ )

Input:  $f \in \mathbb{C}[\mathbf{x}]$ ,  $G = \{g_1, g_2, \dots, g_s\}$ ,  $g_i \in \mathbb{C}[\mathbf{x}]$ ,  $i = 1, 2, \dots, s$ ;  
 $>$  is any monomial ordering.

Output:  $h \in \mathbb{C}[\mathbf{x}] : uf = \sum_{i=1}^s a_i g_i + h.$

$h := f$ ;  $T := G$ ;

while ( $h \neq 0$ ) do

$T_h := \{g \in T : \text{LM}(g) | \text{LM}(h)\}$ ;

    if  $T_h = \emptyset$  then

        return  $h$ ;

    else

        choose  $g \in T_h$  with minimal  $\text{ecart}(g)$ ;

        if  $\text{ecart}(g) > \text{ecart}(h)$  then  $T := T \cup \{h\}$ ;

$h := \text{Spolynomial}(h, g)$ ;

    end if;

end while.

# local membership

## Proposition

Consider  $I \subset \mathbb{C}[\mathbf{x}]$  and  $\mathbf{0} \in \mathbb{C}^n$  is on an irreducible component  $W$  of  $V(I)$ . Let  $G = \{g_1, g_2, \dots, g_s\}$  be a standard basis of  $I$  with respect to some local order  $>$ . Consider  $p \in \mathbb{C}[\mathbf{x}]$  and let  $r = \text{NFMora}(p, G, >)$ . If  $r = 0$ , then  $p \in I(W)$ .

*Proof.* If  $r = 0$ , then  $up = a_1g_1 + a_2g_2 + \dots + a_sg_s$ , where  $u \in \mathbb{C}[\mathbf{x}]$  is invertible in  $\mathbb{C}[\mathbf{x}]_{(\mathbf{x})}$ , i.e.:  $u(0) \neq 0$ .

We have:  $W \subset V(I) \Rightarrow up \in I(W)$ .

Because  $W$  is irreducible ( $I(W)$  is prime):

$$up \in I(W) \Rightarrow u \in I(W) \text{ or } p \in I(W).$$

But since  $u(0) \neq 0$ ,  $u \notin I(W)$ , thus  $p \in I(W)$ . □

# algorithm for a standard basis

Standard( $G, \text{NF}, >$ )

Input:  $G$  is a finite list of polynomials,  
NF is an algorithm to compute a weak normal form  
> is any monomial ordering.

Output:  $S$  is a standard basis for  $\langle G \rangle$ .

```
 $S := G; P := \{ (f, g) \mid f, g \in S, f \neq g \};$   
while  $P \neq \emptyset$  do  
   $(f, g) := \text{pop from } P;$   
   $h := \text{NF}(\text{Spolynomial}(f, g), S, >);$   
  if  $h \neq 0$  then  
     $P := P \cup \{ (h, f) \mid f \in S \};$   
     $S := S \cup \{h\};$   
  end if;  
end while.
```

## the tangent cone

Consider a point  $\mathbf{p} \in \mathbb{C}^n$  and  $f \in \mathbb{C}[x]$ ,  $d = \deg(f)$ .

$f$  as Taylor series about  $\mathbf{p}$ ,  $|\mathbf{a}| = a_1 + a_2 + \cdots + a_n$ ,  $\mathbf{a}! = a_1! a_2! \cdots a_n!$ :

$$\begin{aligned} f(\mathbf{x}) &= f(\mathbf{p}) + \sum_{0 < |\mathbf{a}| \leq d} \frac{1}{\mathbf{a}!} \frac{\partial^{\mathbf{a}} f}{\partial x_1^{a_1} \partial x_2^{a_2} \cdots \partial x_n^{a_n}}(\mathbf{p}) \prod_{i=1}^n (x_i - p_i)^{a_i} \\ &= f_{\mathbf{p},0} + f_{\mathbf{p},1} + \cdots + f_{\mathbf{p},d}. \end{aligned}$$

Define  $f_{\mathbf{p},min}$  as  $f_{\mathbf{p},min}(\mathbf{p}) \neq 0$  and for all  $j < min$ :  $f_{\mathbf{p},j}(\mathbf{p}) = 0$ .

The *tangent cone* of a set  $S$  at  $\mathbf{p}$  is the variety

$$C_{\mathbf{p}}(S) = V(f_{\mathbf{p},min} \mid f \in I(S)).$$

Standard bases compute tangent cones.

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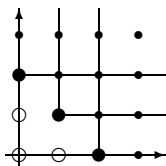
## intersection multiplicity

Let  $\mathbf{z} \in \mathbb{C}^n$  be an isolated solution of  $f(\mathbf{x}) = \mathbf{0}$ .

The intersection multiplicity  $\mu(\mathbf{z})$  is defined algebraically as the dimension of the local quotient ring:

$$\mu(\mathbf{z}) = \dim_{\mathbb{C}} (\mathbb{C}[\mathbf{x}]_{\langle x_1 - z_1, x_2 - z_2, \dots, x_n - z_n \rangle} / \langle f \rangle).$$

In analogy with global quotient rings, consider  $\langle x^2, xy, y^2 \rangle$ :



The big black dots are the generators  $x^2$ ,  $xy$ , and  $y^2$ .

The small black dots are the generated monomials.

The monomials  $1$ ,  $x$ , and  $y$  represented by the empty circles are a basis for the quotient ring. The multiplicity of  $(0, 0)$  equals three, the number of monomials under the staircase.

## using SINGULAR

$$I = \langle x^3 - yz, y^3 - xz, z^3 - xy \rangle,$$

```
> ring R = 32003, (x,y,z), ds;  
> poly f1 = x**3-y*z;  
> poly f2 = y**3-x*z;  
> poly f3 = z**3-x*y;  
> ideal i = f1,f2,f3;  
> ideal j = std(i);  
> j;  
j[1]=xy-z3  
j[2]=xz-y3  
j[3]=yz-x3  
j[4]=x4-z4  
j[5]=y4-z4  
j[6]=z5  
> mult(j);  
11
```

## Using Macaulay 2

We declare a negative lexicographic local order:

```
i1 : R = QQ[x,y,z,MonomialOrder=>
      {Weights=>{-1,0,0},
      Weights=>{0,-1,0},
      Weights=>{0,0,-1}},Global=>>false];

i2 : I = ideal "x3-yz,y3-xz,z3-xy";
o2 : Ideal of R
i3 : I
      3      3      3
o3 = ideal (- y*z + x , y  - x*z, z  - x*y)
o3 : Ideal of R

i4 : B = gens gb I;
      1      8
o4 : Matrix R  <--- R
```



# Milnor and Tjurina numbers

The set of *formal power series* over  $\mathbb{C}$  is

$$\mathbb{C}[[\mathbf{x}]] = \left\{ \sum_{\mathbf{a} \in \mathbb{Z}^n} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \mid c_{\mathbf{a}} \in \mathbb{C} \right\}.$$

The set of *convergent power series* is

$$\mathbb{C}\{\mathbf{x}\} = \left\{ \sum_{\mathbf{a} \in \mathbb{N}^n} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \mid c_{\mathbf{a}} \in \mathbb{C} \right\}.$$

For a singular point at the origin, *the Milnor number*  $\mu$  is

$$\mu = \dim_{\mathbb{C}} \mathbb{C}\{\mathbf{x}\} \left/ \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right\rangle \right.$$

The *Tjurina number*  $\tau$  is

$$\tau = \dim_{\mathbb{C}} \mathbb{C}[[\mathbf{x}]] \left/ \left\langle f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right\rangle \right.$$

Intuitively, the higher the Milnor and Tjurina numbers, the more complicated the structure of the singular point is.

# Summary + Exercises

Standard bases are local analogues to Gröbner bases, used to compute multiplicities.

## Exercises:

- 1 Consider  $I = \langle x_1^2 + x_1^3, x_2^2 \rangle$ .
  - 1 Compute the multiplicity of  $(0, 0)$  by adding a small  $\epsilon$  to both equations. Explain how to derive the multiplicity by looking at the perturbed ideal. What are the conditions for making this calculation work?
  - 2 Verify the multiplicity with Singular.
- 2 Take  $I = \langle f_1 = x_1^3 + x_1 x_2^2, f_2 = x_1 x_2^2 + x_2^3, f_3 = x_1^2 x_2 + x_1 x_2^2 \rangle$ . Use Singular to compute a standard basis for  $I$ . Choose at least two different orderings which lead to different standard bases. Observe that the number of standard monomials is the same for both bases.

## more exercises

- Download the software from <http://www.oliverlabs.net>. Use the Cayley cubic as a benchmark to compare the visualization capabilities of Oliver's software with what Maple and/or Sage has to offer.
- The classification of hypersurface singularities has led to special names:

- ▶  $A_k: f(x_1, x_2, x_3) = x_1^{k+1} + x_2^2 + x_3^2, k \geq 1;$
- ▶  $D_k: f(x_1, x_2, x_3) = x_1 x_2^2 + x_1^{k-1} + x_3^2, k \geq 4;$
- ▶  $E_6: f(x_1, x_2, x_3) = x_1^3 + x_2^4 + x_3^2;$
- ▶  $E_7: f(x_1, x_2, x_3) = x_1^3 + x_1 x_2^3 + x_3^2;$
- ▶  $E_8: f(x_1, x_2, x_3) = x_1^3 + x_2^5 + x_3^2.$

Use software (see the previous exercise) to visualize several instances of these singularities.