Witness Sets

1. Dimension and Degree
   - generic points on solution sets
2. Sevenbar Mechanisms
   - an assembly problem with & without motion
3. A Cascade of Homotopies
   - sequences of embedded polynomial systems
4. Taking Samples of Solution Sets
   - numerical elimination methods
   - a homotopy membership test
5. Theorems of Bézout and Bertini
   - mathematical reflections on the algorithms

MCS 563 Lecture 26
Analytic Symbolic Computation
Jan Verschelde, 14 March 2014

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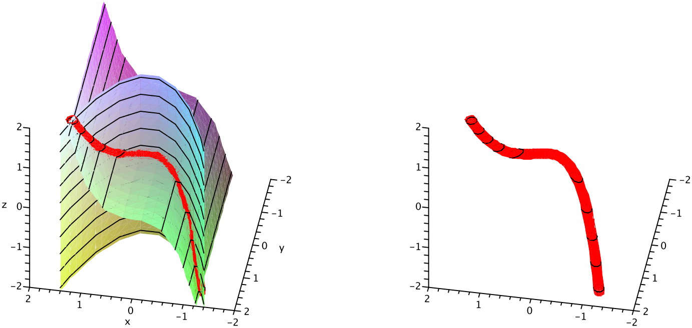
the twisted cubic

$$f(\mathbf{x}) = \left\{ \begin{array}{l} x_2 - x_1^2 = 0 \\ x_3 - x_1^3 = 0 \end{array} \right. \qquad (x_1, x_2, x_3) = (t, t^2, t^3), t \in \mathbb{C}.$$

The twisted cubic is define as the intersection of a quadratic with a cubic surface:



witness sets

To compute the degree of the twisted cubic, consider

$$E(f)(\mathbf{x}) = \left\{ \begin{array}{r} x_2 - x_1^2 = 0 \\ x_3 - x_1^3 = 0 \\ c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 = 0 \end{array} \right. \quad c_0, c_1, c_2, c_3 \in \mathbb{C},$$

where c_0, c_1, c_2, and c_3 are random numbers.
The substitution $x_2 = x_1^2$ and $x_3 = x_1^3$ in the last equation shows that the degree of $f^{-1}(\mathbf{0})$ equals three.

A *witness set* for a k-dimensional solution set consists of

- k hyperplanes with random coefficients; and
- the set of d isolated solutions on those hyperplanes.

Because the hyperplanes are random, all d isolated solutions are generic points and d is the degree of the set.

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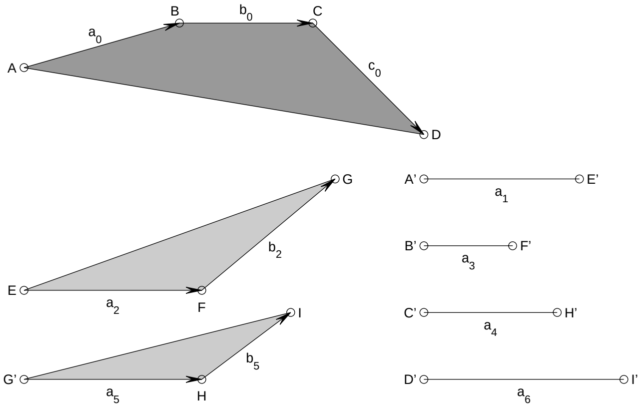
an assembly problem

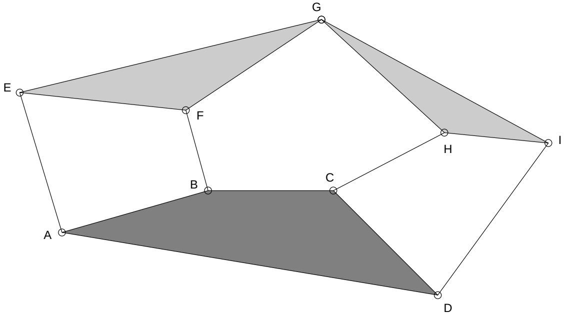
Give



In how many ways can we assemble the pieces?
Which assemblies permit motion? Which ones are rigid?

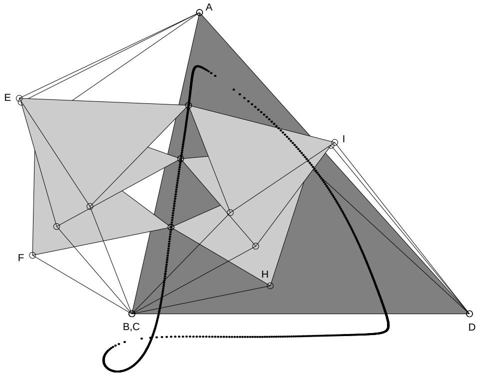
a rigid mechanism

We can view a 7-bar as two 4-bar mechanisms:



a moving mechanism

An exceptional assembly which permits motion is below:



closure equations

A rotation about angle θ is represented by $\Theta = e^{i\theta}$, with $\Theta\overline{\Theta} = 1$.

$$
\left\{
\begin{array}{l}
t_1 T_1 - 1 = 0, t_2 T_2 - 1 = 0, t_3 T_3 - 1 = 0 \\
t_4 T_4 - 1 = 0, t_5 T_5 - 1 = 0, t_6 T_6 - 1 = 0 \\
0.71035834160605 t_1 + 0.46 t_2 - 0.41 t_3 \\
\qquad\qquad + 0.24076130055512 + 1.07248215701824 i = 0 \\
(-0.11 + 0.49 i) t_2 + 0.41 t_3 - 0.502195181179 59 t_4 \\
\qquad\qquad\qquad\qquad\qquad\qquad + 0.41 t_5 = 0 \\
0.50219518117959 t_4 + (-0.09804347826087 \\
\ + 0.43673913043478 i) t_5 - 0.77551855666366 t_6 - 1.2 = 0 \\
0.71035834160605 T_1 + 0.46 T_2 - 0.41 T_3 \\
\qquad\qquad + 0.24076130055512 - 1.07248215701824 i = 0 \\
(-0.11 - 0.49 i) T_2 + 0.41 T_3 \\
\qquad\qquad\qquad - 0.50219518117959 T_4 + 0.41 T_5 = 0 \\
0.50219518117959 T_4 + (-0.09804347826087 \\
-0.43673913043478 i) T_5 - 0.77551855666366 T_6 - 1.2 = 0
\end{array}
\right.
$$

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an example

Consider the system

$$f(\mathbf{x}) = \left\{ \begin{array}{r} (x_1^2 - x_2)(x_1 - 0.5) = 0 \\ (x_1^3 - x_3)(x_2 - 0.5) = 0 \\ (x_1 x_2 - x_3)(x_3 - 0.5) = 0 \end{array} \right.$$

The solutions of the system $f(\mathbf{x}) = \mathbf{0}$ are

- the twisted cubic, a one dimensional solution set; and
- four isolated points.

Can we compute all solutions with one homotopy?

a cascade homotopy

To compute numerical representations of the twisted cubic and the four isolated points, use

$$H(\mathbf{x}, z_1, t) = \left[\begin{array}{ccc} \left[\begin{array}{c} (x_1^2 - x_2)(x_1 - 0.5) \\ (x_1^3 - x_3)(x_2 - 0.5) \\ (x_1 x_2 - x_3)(x_3 - 0.5) \end{array} \right] & + & t \left[\begin{array}{c} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{array} \right] z_1 \\ t\,(c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3) & + & z_1 \end{array} \right] = \mathbf{0}.$$

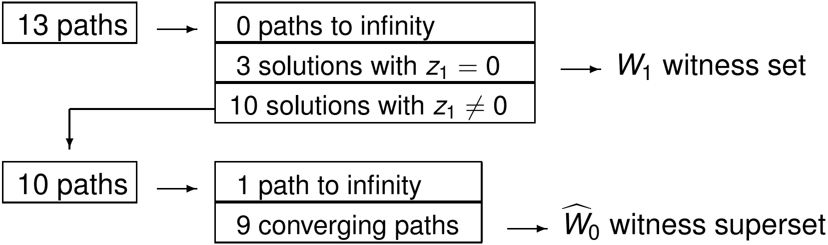
At $t = 1$: $H(\mathbf{x}, z_1, t) = \mathcal{E}_1(f)(\mathbf{x}, z_1) = \mathbf{0}$.

At $t = 0$: $H(\mathbf{x}, z_1, t) = f(\mathbf{x}) = \mathbf{0}$.

As t goes from 1 to 0, the hyperplane is removed from the embedded system, and z_1 is forced to zero.

a superwitness set cascade

Summarizing the progress of the path tracking:



13 paths →

0 paths to infinity

3 solutions with $z_1 = 0$ → W_1 witness set

10 solutions with $z_1 \neq 0$

10 paths →

1 path to infinity

9 converging paths → $\widehat{W}_0$ witness superset

Starting with 13 paths of the embedded system,
the cascade produces three witness points for the cubic
and 9 points which may be isolated or on lie on the cubic.

regularity results

Theorem (superwitness set generation)

For an embedding $\mathcal{E}_i(f)(\mathbf{x}, \mathbf{z})$ of $f(\mathbf{x}) = \mathbf{0}$ with i random hyperplanes and i slack variables $\mathbf{z} = (z_1, z_2, \ldots, z_i)$, we have

1. *solutions with $\mathbf{z} = \mathbf{0}$ contain $\deg W$ generic points on every i-dimensional component W of $f(\mathbf{x}) = \mathbf{0}$;*
2. *solutions with $\mathbf{z} \neq \mathbf{0}$ are regular; and*
3. *the solution paths defined by the cascading homotopy starting at $t = 0$ with all solutions with $z_i \neq 0$ reach at $t = 1$ all isolated solutions of $\mathcal{E}_{i-1}(f)(\mathbf{x}, \mathbf{z}) = \mathbf{0}$.*

an algorithm

```
Input: $f(\mathbf{x}) = \mathbf{0}$ a polynomial system;
       $d$ the top dimension of $f^{-1}(\mathbf{0})$.
Output: $\widehat{W} = [\widehat{W}_d, \widehat{W}_{d-1}, \ldots, \widehat{W}_0]$
        super witness sets for all dimensions.
$V := \mathrm{Solve}(\mathcal{E}_d(f)(\mathbf{x},\mathbf{z}) = \mathbf{0})$;
for $k$ from $d$ down to 1 do
    $\widehat{W}_k := \{ (\mathbf{x},\mathbf{z}) \in V \mid \mathbf{z} = \mathbf{0} \}$;
    $V := \{ (\mathbf{x},\mathbf{z}) \in V \mid z_k \neq 0 \}$;
    if $V = \emptyset$ then return $\widehat{W}$;
    else $h(\mathbf{x},\mathbf{z},t) := (1-t)\mathcal{E}_k(f)(\mathbf{x},\mathbf{z}) + t \left( \begin{array}{c} \mathcal{E}_{k-1}(f)(\mathbf{x},\mathbf{z}) \\ z_k \end{array} \right)$;
         $V := \{ (\mathbf{x},\mathbf{z}) \mid h(\mathbf{x},\mathbf{z},1) = \mathbf{0} \}$;
    end if;
end for;
$\widehat{W}_0 := \{ (\mathbf{x},\mathbf{z}) \in V \mid \mathbf{z} = \mathbf{0} \}$.
```

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Sampling Solutions

Path tracking method implement the following algorithm:

Algorithm SampleSet(W_L,K)

Input: W_L is witness set for k hyperplanes L;
K is a new set of k hyperplanes.
Output: W_K is witness set for hyperplanes K.

The regularity of the paths follows from cheater's homotopy or coefficient-parameter continuation.

numerical elimination methods

By moving the new planes in special position,
we arrive at numerical elimination methods.

Suppose the system $f(\mathbf{x}, \mathbf{y}) = \mathbf{0}$, with $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_k)$ defines a k-dimensional solution set.

Restricting the k hyperplanes in the embedding $\mathcal{E}_k(f)$ to have only nonzero coefficients with the y-variables,
we eliminate the **x**-variables and with SampleSet
we compute points on the projected solution set.

Implementation through the projected samples yields
defining equations for the projection of the solution set.

an example

Let us consider the twisted cubic (once more):

$$\left\{ \begin{array}{rl} y - x^2 = 0 & \\ z - x^3 = 0 & \\ c_0 + c_1 x + c_2 y + c_3 z = 0, & \quad \text{or} \\ c_0 + c_1 x + c_2 y = 0, & \quad \text{or} \\ c_0 + c_1 x + c_3 z = 0. & \end{array} \right.$$

Three choices for linear equations, three cases:

1. The first linear equation gives a cubic in x, substituting y by x^2 and z by x^3 in the linear equation.
2. The substitution for the second choice gives 2 solutions, with $(x, y, z) \mapsto (x, y)$, we sample from $y - x^2 = 0$.
3. For the last linear equation, applying $(x, y, z) \mapsto (x, z)$ leads to sampling $z - x^3 = 0$.

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deciding membership

Given a witness set representation for a solution set, we can decide whether a point belongs to the solution set, via:

Algorithm HomotopyMembershipTest(W_L,$\mathbf{y}$)

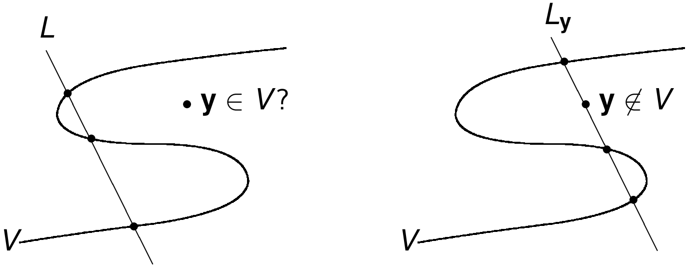
Input: W_L is witness set for a solution set;
$\mathbf{y}$ is any point in space.
Output: yes or no, depending whether $\mathbf{y}$ belongs to the set.

$$h(\mathbf{x}, t) = (1 - t) \left(\begin{array}{c} f(\mathbf{x}) = \mathbf{0} \\ L(\mathbf{x}) = \mathbf{0} \end{array} \right) + t \left(\begin{array}{c} f(\mathbf{x}) = \mathbf{0} \\ L(\mathbf{x}) = L(\mathbf{y}) \end{array} \right) = \mathbf{0};$$

$V := \{ \mathbf{x} \mid h(\mathbf{x}, 1) = \mathbf{0} \}$;
return $\mathbf{y} \in V$.

schematic membership

A curve V is represented by 3 witness points on L:



To decide whether $\mathbf{y} \in V$,
we create a new witness set for a line $L_{\mathbf{y}}$ through $\mathbf{y}$.

As $\mathbf{y} \notin V \cap L_{\mathbf{y}}$, we conclude $\mathbf{y} \notin V$.

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Refining Bézout's Theorem

The correctness of the cascade of homotopies implies

Theorem (refined version of Bézout's theorem)

For a system $f(\mathbf{x}) = \mathbf{0}$ *of* N *polynomials* $f = (f_1, f_2, \ldots, f_N)$ *in* n *unknowns* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$*, the total degree*

$$D = \prod_{i=1}^{N} \deg(f_i) \geq \sum_{j=0}^{n} \mu_j \deg(W_j),$$

bounds the degree of all j*-dimensional solution sets* W_j*, each properly counted with their multiplicity* μ_j*.*

To prove the correctness we need Bertini's theorem.

Bertini's theorem

The generic element of a linear system is smooth away from the base locus of the system.

Theorem (Bertini's Theorem)

Let $f = (f_1, f_2, \ldots, f_N)$ *be a tuple of* N *polynomials representing a collection of* N *hypersurfaces* $f_i^{-1}(\mathbf{0})$ *in* $\mathbb{C}^n$*, for* $i = 1, 2, \ldots, N$*. Consider*

$$g = \lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda f_N, \quad \lambda_i \in \mathbb{C}, i = 1, 2, \ldots, N.$$

Then, for generic choices of λ_i*:*

$$\frac{\partial g}{\partial x_j}(\mathbf{z}) \neq 0, \ \textit{for all } \mathbf{z} \notin \bigcap_{i=1}^{N} f^{-1}(\mathbf{0}).$$

sketch of proof

It suffices to show the theorem for two hypersurfaces: p and q and to consider $p(\mathbf{x}) + \lambda q(\mathbf{x}) = 0$.

Let $\mathbf{z}$ be a singular point outside the base locus, in particular: $q(\mathbf{z}) \neq 0$.

We must show that singular points can only happen for finitely many values of λ.

We then have that $\mathbf{z}$ satisfies the system

$$\left\{ \begin{array}{l} p(\mathbf{x}) + \lambda q(\mathbf{x}) = 0 \\ \dfrac{\partial p}{\partial x_i} + \lambda \dfrac{\partial q}{\partial x_i} = 0, \quad i = 1, 2, \ldots, n, \end{array} \right.$$

which defines an algebraic set $V \subset \mathbb{C}^n \times \mathbb{C}$.

sketch continued

At $\mathbf{z}$ we have $\lambda = -\dfrac{p(\mathbf{z})}{q(\mathbf{z})}$ and therefore:

$$\frac{\partial p}{\partial x_i}(\mathbf{z}) - \frac{p(\mathbf{z})}{q(\mathbf{z})}\frac{\partial q}{\partial x_i}(\mathbf{z}) = 0, \quad \text{for } i = 1, 2, \ldots, n.$$

Consider

$$\frac{\partial}{\partial x_i}\left(\frac{p}{q}\right) = \frac{\dfrac{\partial p}{\partial x_i} q - p \dfrac{\partial q}{\partial x_i}}{q^2} = \frac{\dfrac{\partial p}{\partial x_i} - \dfrac{p}{q}\dfrac{\partial q}{\partial x_i}}{q}.$$

At $\mathbf{z}$: $\dfrac{\partial}{\partial x_i}\left(\dfrac{p}{q}\right) = 0$. So at singular points $\dfrac{p}{q}$ is constant.

But singular points belong to the algebraic set V and p/q can only be constant over finitely many points. ☐

Summary + Exercises

Witness sets belong to numerical algebraic geometry.

Exercises:

1. Consider

$$f(x_1, x_2, x_3) = \left\{ \begin{array}{r} x_1^2 + x_2^2 - 1 = 0 \\ x_2^2 + x_3^2 - 1 = 0 \\ c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 = 0 \end{array} \right.$$

   where c_0, c_1, c_2, and c_3 are random numbers.

   1. Describe geometrically the solution set of this polynomial system, considering the last equation as a moving plane to sweep the curve defined by the intersection of the first two equations. Use Maple or Sage to illustrate the geometry. Can you see the degree of the curve defined by the first two equations?
   2. Explain what happens when $c_3 = 0$. Give a geometric interpretation of this special choice of hyperplane. Likewise, show that setting $c_2 = 0$ corresponds to projecting the solution set onto the (x_1, x_3)-plane.

more exercises

2. Solve the sevenbar system as follows. Use `phc -c` (option #1) to construct $\mathcal{E}(f)$, solve $\mathcal{E}(f) = \mathbf{0}$ with `phc -b`, and then call `phc -c` again to run the cascade. Check the output to verify whether a witness set for the curve has six points and check the size of the witness superset $\widehat{W}_0$. Count the paths that have been tracked.
3. Consider the systems

$$f(x,y,z) = \left\{ \begin{array}{l} y - x^2 = 0 \\ z - x^3 = 0 \end{array} \right. \qquad g(x,y,z) = \left\{ \begin{array}{l} xy - z^2 = 0 \\ xz - y^2 = 0. \end{array} \right.$$

   1. Construct witness sets for both $f^{-1}(\mathbf{0})$ and $g^{-1}(\mathbf{0})$.
   2. Take a generic point from $f^{-1}(\mathbf{0})$ and test whether it belongs to $g^{-1}(\mathbf{0})$ and vice versa.
   3. Do tests like these allow you to decide whether $f^{-1}(\mathbf{0}) \subseteq g^{-1}(\mathbf{0})$ or $g^{-1}(\mathbf{0}) \subseteq f^{-1}(\mathbf{0})$? Explain.