Tropical Curves and Amoebas

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Graduate Computational Algebraic Geometry Seminar

Tropical Curves and Amoebas

1. Introduction
   - Introduction to Tropical Geometry
2. Tropical Curves
   - the zero set of a plane tropical curve
   - the balancing condition
   - Bézout's theorem for tropical curves
   - stable intersection
3. Amoebas and their Tentacles
   - the amoeba of a Laurent polynomial ideal

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Introduction to Tropical Geometry

Introduction to Tropical Geometry is the title of a forthcoming book of Diane Maclagan and Bernd Sturmfels.

The web page
`http://homepages.warwick.ac.uk/staff/D.Maclagan/papers/TropicalBook.html`
offers the pdf file of the first five chapters (23 August 2013).

Tropical islands is the title of the first chapter,
which promises a friendly welcome to tropical mathematics.

Today we look at sections 1.3 and 1.4.

overview of the book

The titles of the five chapters with some important sections:

1. Tropical Islands
    - amoebas and their tentacles
    - implicitization
2. Building Blocks
    - polyhedral geometry
    - Gröbner bases
    - tropical bases
3. Tropical Varieties
    - the fundamental theorem
    - the structure theorem
    - multiplicities and balancing
    - connectivity and fans
    - stable intersection
4. Tropical Rain Forest
5. Linear Algebra

Tropical Curves and Amoebas

plane tropical curves

Definition

The *tropical semiring* is denoted as $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$, with

$$x \oplus y = \min(x, y) \quad \text{and} \quad x \odot y = x + y.$$

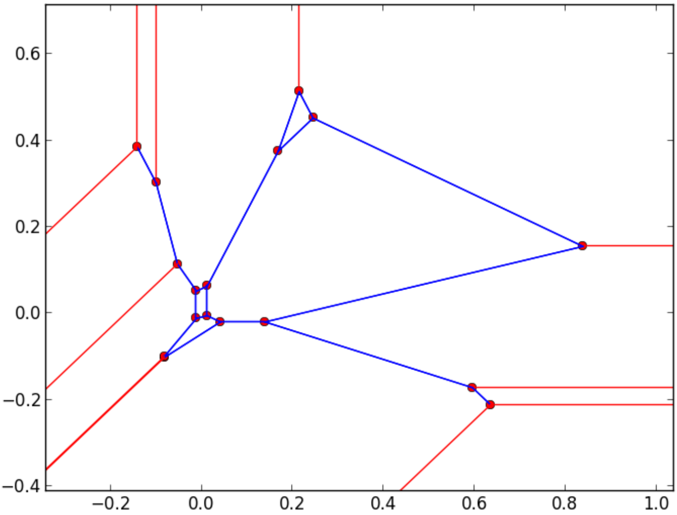
A plane tropical curve is the zero set of a tropical polynomial in two variables. A quadratic tropical polynomial

$$p(x, y) = c_{2,0} \odot x^{\odot 2} \oplus c_{1,1} \odot x \odot y \oplus c_{0,2} \odot y^{\odot 2} \oplus c_{1,0} \odot x \oplus c_{0,1} \odot y \oplus c_{0,0}$$

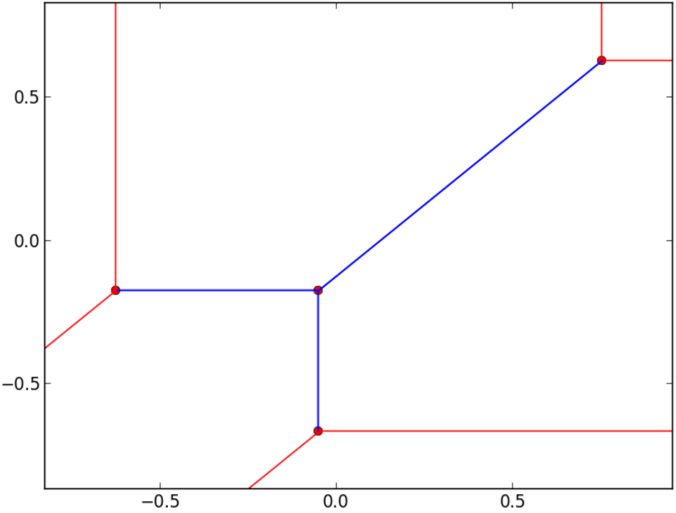
defines the function

$$f(x, y) = \min(c_{2,0} + 2x, c_{1,1} + x + y, c_{0,2} + 2y, c_{1,0} + x, c_{0,1} + y, c_{0,0}).$$

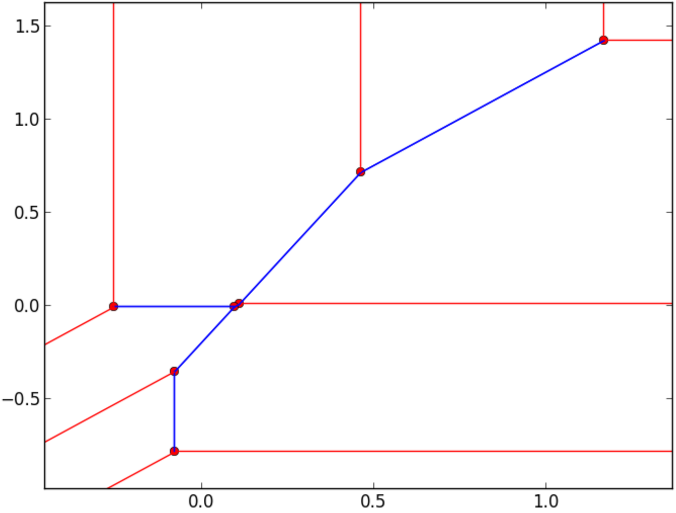
plot of a plane tropical curve of degree ten



plot of a plane tropical curve of degree two



plot of a plane tropical curve of degree three



plotting a plane tropical curve

Consider plane tropical curves with random real coefficients $c_{i,j} \in [-1, +1]$:

$$f(x,y) = \bigoplus_{0 \leq i+j \leq d} c_{i,j} \odot x^{\odot i} \odot y^{\odot j} = \min_{0 \leq i+j \leq d} \left(c_{i,j} + ix + jy \right).$$

Taking advantage of the randomness of the coefficients,
the zero set of a plane tropical curve consists of

1. nodes: where the minimum is attained three times:

$$\left\{ \begin{array}{l} \min(c_{i_1,j_1} + i_1 x + j_1 y) = \min(c_{i_0,j_0} + i_0 x + j_0 y) \\ \min(c_{i_2,j_2} + i_2 x + j_2 y) = \min(c_{i_0,j_0} + i_0 x + j_0 y) \end{array} \right.$$

2. segments between two nodes;
3. half rays starting at nodes.

computing all nodes

At a node, the minimum is attained three times:

$$\left\{ \begin{array}{rcl} \min(c_{i_\ell,j_\ell} + i_\ell x + j_\ell y) & = & \min(c_{i_k,j_k} + i_k x + j_k y) \\ \min(c_{i_m,j_m} + i_m x + j_m y) & = & \min(c_{i_k,j_k} + i_k x + j_k y) \end{array} \right.$$

1. For the list of all exponents, enumerate all triplets (k, ℓ, m), with $k < \ell < m$.
2. For each triplet (k, ℓ, m), solve the linear system

$$\left\{ \begin{array}{rcl} (i_\ell - i_k)x + (j_\ell - j_k)y & = & c_{i_k,j_k} - c_{i_\ell,j_\ell} \\ (i_m - i_k)x + (j_m - j_k)y & = & c_{i_k,j_k} - c_{i_m,j_m} \end{array} \right.$$

3. For each solution (x, y) check whether the minimum is indeed attained three times.

segments and half rays

For any pair of two distinct nodes:

1. Compute the mid point of the coordinates of the nodes.
2. If at the mid point, the minimum is attained twice,
   then draw a segment between the two nodes.

Any node is defined by a triplet (k, ℓ, m) with three lines $c_{i_k,j_k} + i_k x + j_k y$, $c_{i_\ell,j_\ell} + i_\ell x + j_\ell y$, and $c_{i_m,j_m} + i_m x + j_m y$.

To draw the rays, at any node:

1. Compute the direction of the intersection
   between any pair of the three lines that pass through the node.
2. If the minimum in that direction is attained twice,
   then draw a half ray in that direction.

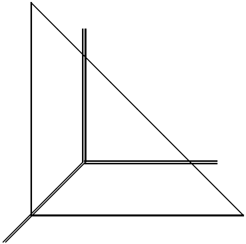
inner normals to edges of Newton polygons

For any plane curve of degree d: we have at most d half rays pointing in each direction, east, nord or southwest.

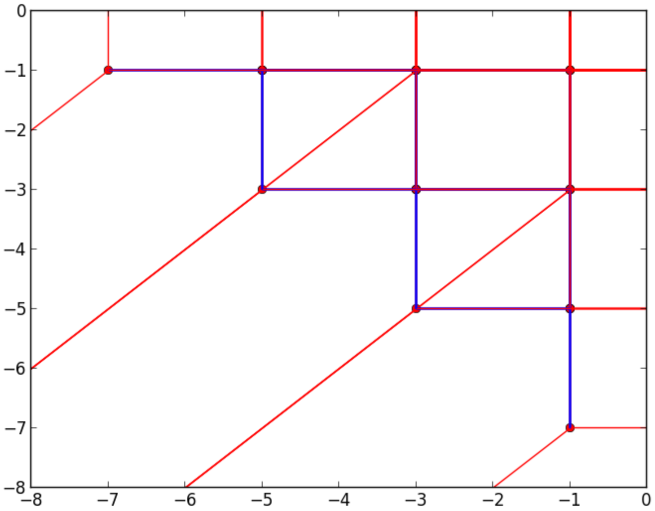
The rays are inner normals to the edges of the Newton polygon:

1. The half rays pointing east $\perp$ vertical edge, $\mathrm{conv}(\{(0,0),(0,d)\})$.
2. Those pointing north $\perp$ horizontal edge, $\mathrm{conv}(\{(0,0),(d,0)\})$.
3. Those pointing southwest $\perp$ slanted edge, $\mathrm{conv}(\{(d,0),(0,d)\})$.

The Newton polygon of a dense polynomial is a triangle:



a fine curve of degree four with $c_{i,j} = i^2 + j^2$



regular subdivisions

The cells in a regular subdivision of a polygon are in one-to-one correspondence with the facets on the lower hull of the lifted polygon.

A tropical curve $V(p)$ is a planar graph dual to the graph of a regular subdivision of the Newton polygon of p:

- A node in the zero set of the tropical curve corresponds to a cell in the subdivision.
- A segment between two nodes is perpendicular to the edge between the two cells in the subdivision.
- A half ray is perpendicular to an edge of the Newton polygon.

The coefficients of the tropical polynomial are related to the heights in the lifting function.

the balancing condition

Proposition

The curve $V(p)$ of a tropical polynomial p is a finite graph which is embedded in the plane $\mathbb{R}^2$.

- *The graph has both bounded and unbounded edges,*
- *all edge slopes are rational, and*
- *it satisfies a balancing condition around each node.*

Balancing refers to the following geometric condition.

Consider any node (x, y) of the graph and suppose it is $(0, 0)$.

Then the edges adjacent to this node lie on lines with rational slopes.

On each ray emanating from $(0, 0)$ take the first nonzero lattice vector.

Balancing at (x, y) means that a weighted sum of these vectors is zero, where the weights are fixed for each edge.

intersecting tropical curves

The following statements are true:

- Two general lines meet in one point.
- Two general points lie on a unique line.
- A general line and quadric meet in two points.
- Two general quadrics meet in four points.
- Five general points lie on a unique quadric.

Observe the word *general*.

intersection multiplicity

Every edge of a tropical curve has an attached positive integer which is its multiplicity:

- For any point in the relative interior of any edge, make the sum of all terms on that edge to form a polynomial.
- The number of nonzero roots of that polynomial equals the lattice length of the edge in question.

Definition

Two tropical curves *intersect transversally* if every common point lies in the relative interior of a unique edge of both curves.

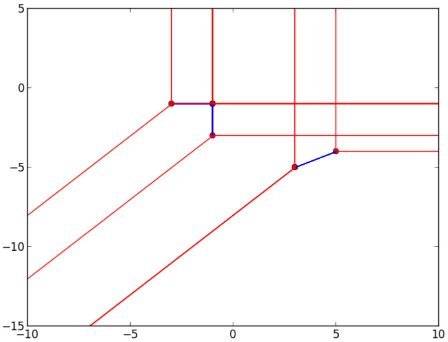
Definition

Suppose two edges intersect transversally and their primitive direction vectors are $(u_1, u_2) \in \mathbb{Z}^2$ and $(v_1, v_2) \in \mathbb{Z}^2$. The *intersection multiplicity* of the intersection point is then the determinant $|u_1 v_2 - u_2 v_1|$.

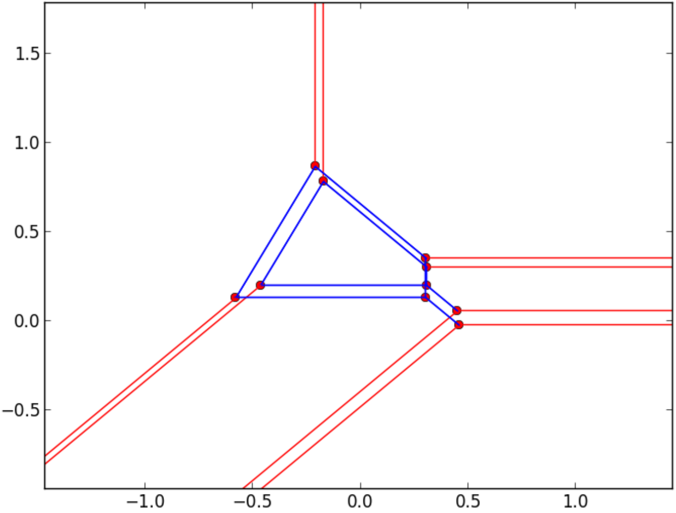
Bézout's theorem for tropical curves

Theorem (Bézout)

Consider two plane tropical curves C and D of degree c and d. If the two curves intersect transversally, then then the number of intersection points, counted with multiplicities, is $c \cdot d$.



stable intersection of a cubic with itself



the stable intersection principle

We can remove the *intersect transversally* assumption in the theorem of Bézout by considering curves C_ϵ and D_ϵ that are near to the original curves C and D.

Theorem (stable intersection principle)

The limit of $C_\epsilon \cap D_\epsilon$ is independent of the choice of ϵ.

By this stable intersection principle, we can define the following.

Definition

The *stable intersection* of two curves C and D is

$$C \cap_{\rm st} D = \lim_{\epsilon \rightarrow 0} (C_\epsilon \cap D_\epsilon),$$

where multiplicities of colliding points are added.

This leads to a stronger version of Bézout's theorem.

Tropical Curves and Amoebas

the amoeba of an ideal

Let I be an ideal in the Laurent polynomial ring $\mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Because we allow negative exponents, denote $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

The variety of I is the common zero set of all $f \in I$:

$$V(I) = \{ \, \mathbf{z} \in (\mathbb{C}^*)^n \; : \; f(\mathbf{z}) = 0 \text{ for all } f \in I \, \}.$$

Apply the coordinate wise logarithmic map to $V(I)$:

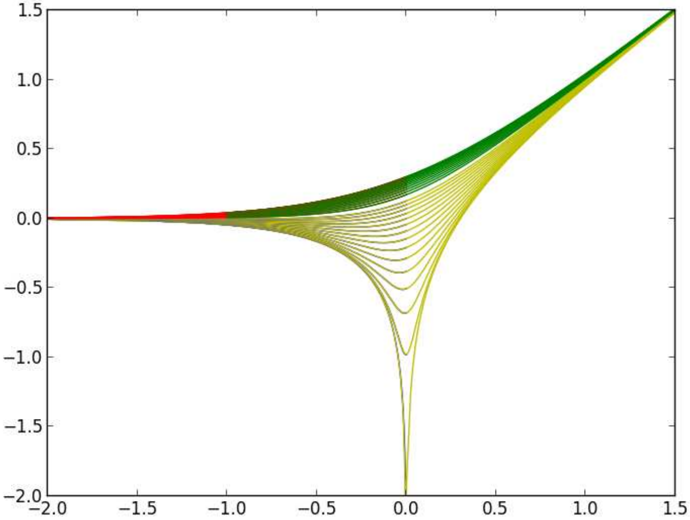
Definition

For an ideal I in $\mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$, the *amoeba* of I is

$$\mathcal{A}(I) = \{ \, (\log(|z_1|, \log(|z_2|), \ldots, \log(|z_n|)) \; : \; (z_1, z_2, \ldots, z_n) \in V(I) \, \}.$$

Introduced by Gel'fand, Kapranov, and Zelevinsky in *Discriminants, Resultants, and Multidimensional Determinants*, 1994.

the amoeba of $x + y = 1$



compactifying the amoeba

The plot of the amoeba for $x + y = 1$ used polar coordinates:
$x = r\exp(i\theta)$ for a range of values for r and θ.

We compactify the amoeba of $f^{-1}(0)$:

- Take lines perpendicular to the tentacles.
- As each line cuts the plane in half, keep those halves of the plane where the amoeba lives.

The intersection of all half planes defines a polygon.

The resulting polygon is the Newton polygon of f.