

Amalgamation, Absoluteness, and Categoricity

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Abstract

We describe the major result on categoricity in $L_{\omega_1, \omega}$, placing it in the context of more general work in abstract elementary classes. In particular, we illustrate the role of higher dimensional amalgamations and sketch the role of a weak extension of ZFC in the proof. We expound the translation of the problem to studying atomic models of first order theories. We provide a simple example of the failure of amalgamation for a complete sentence of $L_{\omega_1, \omega}$. We prove some basic results on the absoluteness of various concepts in the model theory of $L_{\omega_1, \omega}$ and publicize the problem of absoluteness of \aleph_1 -categoricity in this context. Stemming from this analysis, we prove Theorem: The class of countable models whose automorphism groups admit a complete left invariant metric is Π_1^1 but not Σ_1^1 .

The study of infinitary logic dates from the 1920's. Our focus here is primarily on the work of Shelah using stability theoretic methods in the field (beginning with [She75]). In the first four sections we place this work in the much broader context of abstract elementary classes (aec), but do not develop that subject here. The main result discussed, Shelah's categoricity transfer theorem for $L_{\omega_1, \omega}$ explicitly uses a weak form of the GCH. This raises questions about the absoluteness of fundamental notions in infinitary model theory. Sections 5-7 and the appendix due to David Marker describe the complexity and thus the absoluteness of such basic notions as satisfiability, completeness, ω -stability, and excellence.¹ We state the question, framed in this incisive way by Laskowski, of the absoluteness of \aleph_1 -categoricity. And from the model theoretic characterization of non-extendible models we derive the theorem stated in the abstract on the complexity of automorphism groups. Most of the results reported here in Sections 1-4 are due to Shelah; the many references to [Bal09] are to provide access to a unified exposition. I don't know anywhere that the results in Section 5 have been published; but the techniques are standard and our goal is just to provide a reference. The result in Section 6 is new but easy.

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1 The Universe is Wide or Deep

Shelah made the following rough conjecture: Let \mathcal{K} be a *reasonable* class of models.

Either for some λ , there are many models of cardinality λ or there are models of arbitrarily large cardinality.

Our metaphor requires some explanation. ‘The universe’ should perhaps be ‘each universe’; universe refers to all models in a specific class. Further we are taking ‘or’ in the inclusive sense. Certainly, there are classes (e.g. dense linear orders) which are both wide and deep. Perhaps, taking narrow, as meaning there are few models in each cardinality, the aphorism better reads. A narrow universe is deep. It turns out that this question depends very much on the choice of ‘reasonable’. It also seems to be sensitive to the choice of axioms of set theory. In order to give a precise formulation of the conjecture we have to specify ‘many’ and the notion of a ‘reasonable class’. In general ‘many’ should mean 2^λ ; but in important cases that have been proved, it is slightly smaller.

As is often the case there are some simplifying assumptions in this area that have been internalized by specialists but obscure the issues for other logicians. We try to explain a few of these simplifications and sketch some of the major results.

Some historical background will help clarify the issues. Much model theoretic research in the 60’s focussed on general properties of first order and infinitary logic. A number of results seemed to depend heavily on extensions of ZFC. For example, both Keisler’s proof that two structures are elementarily equivalent if and only if they have isomorphic ultrapowers and Chang’s proof of two cardinal transfer required GCH. In general, even the existence of saturated models depends on the GCH. Shelah removed the set-theoretic hypothesis from Keisler’s theorem. But various versions of two cardinal transfer were proven to require GCH and even large cardinal hypotheses. See [CK73].

The invention of stability theory radically recast the subject of model theory. E.g., for various classes in the stability hierarchy, it is straightforward to characterize in ZFC exactly in which cardinals there are saturated models. And for the best behaved theories the answer is: all cardinals. Further, for countable stable theories Shelah and Lachlan independently showed that two cardinal transfer between any pair of cardinalities is true in ZFC. Moreover, the fundamental notions of first order stability theory are absolute.

For first order logic, our guiding question is trivial². If a theory has an infinite model then it has arbitrarily large models. The question is interesting for theories in logics which fail the upward Löwenheim-Skolem theorem. The notion of an Abstract Elementary Class (AEC) provides a general framework for analyzing such classes. But as we show in the next section the conjecture is trivially false in that case. It is not too difficult to find in ZFC examples (Example 2.1) of AEC that have no model above \aleph_1 but that are \aleph_1 -categorical [She09a, Bal09]. And in $L_{\omega_1, \omega}(Q)$, it is consistent (via Martin’s axiom) that there are \aleph_1 -categorical sentences with no model of cardinality greater than 2^{\aleph_0} . But those sentences have many models in 2^{\aleph_0} . In this note we describe how for $L_{\omega_1, \omega}$, there are major advances on the target problem. They use extensions of ZFC

²The main gap theorem, every first order theory either eventually has the maximal number of models or the number of models is bounded by a small function, has the same flavor. And in fact the argument for this result arose after Shelah’s consideration of the infinitary problems.

but rather mild ones; the initials below refer to the ‘Weak Continuum Hypothesis’ and the ‘Very Weak Continuum Hypothesis’:

WGCH: Cardinal exponentiation is an increasing function.

VWGCH: Cardinal exponentiation is an increasing function below \aleph_ω .

This leaves us with two more precise questions.

1. Does the proof of the conjecture for $L_{\omega_1, \omega}$ (see Section 4) really need VWGCH?
2. Is the conjecture ‘eventually true’ for AEC’s³?

Much of core mathematics studies either properties of particular structures of size at most the continuum or makes assertions that are totally cardinal independent. E.g., if every element of a group has order two then the group is abelian. Model theory and even more clearly infinitary model theory allows the investigation of ‘structural properties’ that are cardinal dependent such as: existence of models, spectra of stability, and number of models and existence of decompositions. Often these properties can be tied to global conditions such as the existence of a ‘good’ notion of dependence.

2 Abstract Elementary Classes

We begin by discussing the notion of an abstract elementary class. The examples show that this is too broad a class to be ‘reasonable’ for our target problem. But some positive results can be proved in this general setting; this generality exposes more clearly what is needed for the argument by avoiding dependence on accidental syntactic features.

An abstract elementary class $(\mathbf{K}, \prec_{\mathbf{K}})$ ⁴ is a collection of structures for a fixed vocabulary τ that satisfy, where $A \prec_{\mathbf{K}} B$ means in particular A is a substructure of B ,

1. If $A, B, C \in \mathbf{K}$, $A \prec_{\mathbf{K}} C$, $B \prec_{\mathbf{K}} C$ and $A \subseteq B$ then $A \prec_{\mathbf{K}} B$;
2. Closure under direct limits of $\prec_{\mathbf{K}}$ -embeddings;
3. Downward Löwenheim-Skolem. If $A \subset B$ and $B \in \mathbf{K}$ there is an A' with $A \subseteq A' \prec_{\mathbf{K}} B$ and $|A'| \leq |A| + \text{LS}(\mathbf{K})$.

The invariant $\text{LS}(\mathbf{K})$, is a crucial property of the class. The class of well-orderings satisfies the other axioms (under end extension) but is not an AEC.

Two easy examples are: First order and $L_{\omega_1, \omega}$ -classes; $L(Q)$ classes have Löwenheim-Skolem number \aleph_1 . For the second case one has to be careful about the definition of $\prec_{\mathbf{K}}$ – being an $L(Q)$ -elementary submodel does not work (a union of a chain can make $(Qx)\phi(x)$ become true even if it is false along the chain).

The notion of AEC has been reinterpreted in terms of category theory by Kirby: “Abstract Elementary Categories” [Kir08] and by Lieberman: “AECs as accessible categories” [Lie].

It is easy to see that just AEC is too weak a condition for the general conjecture.

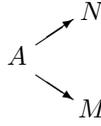
³For much positive work in this direction see [She09a].

⁴Naturally we require that both \mathbf{K} and $\prec_{\mathbf{K}}$ are closed under isomorphism.

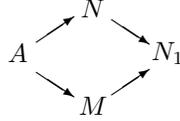
Example 2.1 The class of well-orderings with order-type at most ω_1 with $\prec_{\mathbf{K}}$ as initial segment is an AEC with \aleph_1 countable models. It is \aleph_1 -categorical and satisfies both amalgamation and joint embedding but is not ω -Galois stable [Kue08]. And in fact there is no model of cardinality \aleph_2 . So this universe is neither wide nor deep.

Let's clarify the specific meaning of the amalgamation property in this context.

Definition 2.2 *The class \mathbf{K} satisfies the amalgamation property if for any situation with $A, M, N \in \mathbf{K}$:*



there exists an $N_1 \in \mathbf{K}$ such that



Note that we have required the base structure A to be in \mathbf{K} ; this is sometimes referred to as ‘model amalgamation’. Requiring amalgamation over arbitrary substructures A is a much stronger condition, which fails for important natural examples such as Zilber’s pseudo-exponential field [Zil04]. There is much work in homogenous model theory where the stronger homogeneity condition is assumed.

The existence of amalgamations is an absolutely fundamental problem for AEC and for any study of infinitary logic. In first order logic it is easy to show that for complete theories amalgamation always holds over models with \prec as elementary extension. And it holds over arbitrary subsets of models if T admits elimination of quantifiers. Here is a basic example of failure for a complete sentence of $L_{\omega_1, \omega}$.

Example 2.3 Let T be the first order theory in a language with binary relation symbols $\langle E_i : i < \omega \rangle$ that asserts the E_i are infinitely many refining equivalence relations with binary splitting.

Using $L_{\omega_1, \omega}$ the equivalence relation E_∞ , the intersection of the given equivalence relations, is definable. Add two unary predicates (blue and red) and the infinitary axioms

1. Each E_∞ -class contains infinitely many elements.
2. Every element of an E_∞ -class is red or every element is blue.
3. Blue and red divide the E_∞ -classes into dense and codense sets.

Now it is easy to check that these axioms are \aleph_0 -categorical but fail amalgamation (since a new path may be either red or blue).

We introduced the notion of abstract elementary class in this paper in order to state One Completely General Result which can be found in I.3.8 of [She09a] or [She83b, Bal09].

Theorem 2.4 (WGCH) *Suppose $\lambda \geq \text{LS}(\mathbf{K})$ and \mathbf{K} is λ -categorical. If amalgamation fails in λ there are 2^{λ^+} models in \mathbf{K} of cardinality λ^+ .*

As opposed to many other results in the study of abstract elementary classes which rely on an additional collection of model theoretic hypotheses, this result is about *all* AEC's. Moreover, variants of the proposition recur repeatedly in the proof of the main result being expounded. The argument uses weak diamond and is primarily combinatorial; it proceeds directly from the definition of an AEC. The result fails under $MA + \neg CH$. An example is presented in [She87, She09a] and a simpler one in [Bal09]. It is an AEC (even given by a theory in $L(Q)$) which fails amalgamation in \aleph_0 , but becomes \aleph_1 -categorical in a forcing extension. But it remains open whether there are such examples in $L_{\omega_1, \omega}$. Easy examples ([BKS09]) show the categoricity is a necessary condition for Theorem 2.4. This has a fundamental impact on the structure of the main proof. Because of this we must pass to complete sentences and gain categoricity in \aleph_0 . Shelah's approach through frames in [She09a] aims at weakening the need for categoricity at the cardinal where the induction commences.

Amalgamation plays a fundamental role in the study of AECs. One line of research pioneered by Shelah [She99] and highly developed by Grossberg, VanDieren, and Lessmann in a series of papers (e.g [GV06]) assumes both arbitrarily large models and amalgamation; under strong model theoretic assumptions the results are proved in ZFC. An account of this work with full references to the published papers appears in Part II of [Bal09]. In this paper we focus on earlier work on $L_{\omega_1, \omega}$, which is a little more concrete as the logic is fixed. But it is more general in another way. Rather than assuming amalgamation, failure of amalgamation is shown to create width. Both amalgamation and the existence of large models are proved for narrow classes; this brings the set theoretic difficulties into view. The work of Hyttinen and Kesala on finitary AEC (e.g [HK07]) continues the program of assuming arbitrarily large models and amalgamation. But, even stronger model theoretic assumptions lead to the development of a geometric stability theory. Several further directions of study in AEC are explored in [She09a]. The introduction to that book surveys the field and explains Shelah's viewpoint. The method of frames, expounded in [She09a], provides an approach to the problem of building larger models from categoricity in one or several successive uncountable cardinals; he attempts to avoid the traces of compactness that simplify the work starting at \aleph_0 and \aleph_1 in $L_{\omega_1, \omega}$. In other papers Shelah (e.g [She01]) considers the general problem of eventual categoricity assuming large cardinal axioms.

3 From $L_{\omega_1, \omega}$ to first order

We begin by translating the problem from infinitary logic into the study of specific subclasses of models of first order theories. This removes the distraction of developing

new notions of each syntactic idea (e.g. type) for each fragment of $L_{\omega_1, \omega}$. More subtly, for technical reasons we need to restrict to complete sentences in $L_{\omega_1, \omega}$. (This restriction to complete sentences is automatic in the first order case but its legitimacy is only proved in certain cases for infinitary logic).

Definition 3.1 For Δ a fragment of $L_{\omega_1, \omega}$, a Δ -theory T is Δ -complete if for every Δ -sentence ϕ , $T \models \phi$ or $T \models \neg\phi$. We may write complete when $\Delta = L_{\omega_1, \omega}$.

Definition 3.2 1. A model M of a first order theory is called atomic if each finite sequence from M realizes a principal type over the empty set – one generated by a single formula.

2. An atomic class is an aec, consisting of the atomic models of a complete first order theory T with elementary submodel as the notion of strong submodel. \mathbb{M} is a large saturated model of T ; it is usually not atomic. A set $A \subset \mathbb{M}$ is atomic set if each finite sequence from A realizes a principal type over the empty set -generated by a single formula.

The study of categoricity (at least from \aleph_1 upwards), in $L_{\omega_1, \omega}$ can be translated to the study of atomic models of a first order theory. This is non-trivial. The argument begins with a fundamental result from the early 60's.

Theorem 3.3 (Chang/Lopez-Escobar) Let ψ be a sentence in $L_{\omega_1, \omega}$ in a countable vocabulary τ . Then there is a countable vocabulary τ' extending τ , a first order τ' -theory T , and a countable collection of τ' -types Γ such that reduct is a 1-1 map from the models of T which omit Γ onto the models of ψ .

The proof is straightforward. E.g., for any formula ψ of the form $\bigwedge_{i < \omega} \phi_i$, add to the language a new predicate symbol $R_\psi(\mathbf{x})$. Add to T the axioms

$$(\forall \mathbf{x})[R_\psi(\mathbf{x}) \rightarrow \phi_i(\mathbf{x})]$$

for $i < \omega$ and omit the type $p = \{\neg R_\psi(\mathbf{x})\} \cup \{\phi_i : i < \omega\}$.

Thus we have restricted to the models of a theory that omit a family Γ of types, but that may realize some non-principal types. Shelah observed that if T had only countably many types then a similar expansion of the vocabulary gives a T' such that the required interpretation is obtained by omitting *all* non-principal types. That is, the object of study is the atomic models of T' . This further reduction is technically important. In particular it implies ω -categoricity.

But why can we assume that the T associated with ϕ has only countably many types over the empty set? We need a few definitions to give an explanation.

Definition 3.4 Fix a sentence $\phi \in L_{\omega_1, \omega}$ and let Δ be a countable fragment of $L_{\omega_1, \omega}$ containing ϕ .

1. A τ -structure M is Δ -small if M realizes only countably many Δ -types (over the empty set).

2. An $L_{\omega_1, \omega}$ -sentence ϕ is Δ -small if there is a set X countable of complete Δ -types over the empty set and each model realizes some subset of X .

'small' means $\Delta = L_{\omega_1, \omega}$

It is easy to see that if M is small then M satisfies a complete sentence. If ϕ is small then Scott's argument for countable models generalizes and there is a complete sentence ψ_ϕ such that: $\phi \wedge \psi_\phi$ has a countable model. So ψ_ϕ implies ϕ . But ψ_ϕ is not in general unique. For example ϕ might be just the axioms for algebraically closed fields. Two choices for ψ_ϕ are the Scott sentence of the prime field and the Scott sentence for the model of transcendence degree \aleph_0 . Only the second has an uncountable model.

We can make an appropriate choice of ψ_ϕ if ϕ is \aleph_1 -categorical. There are two ingredients in the choice.

Theorem 3.5 (Shelah) *If ϕ has an uncountable model M that is Δ -small for every countable Δ and ϕ is κ -categorical then ϕ is implied by a complete sentence ψ with a model of cardinality κ .*

This result appears first in [She83a]. It is retold in [Bal09]; in [Bal07], we adapt the argument to give a model theoretic proof of a result of Makkai (obtained by admissible set theory) that a counterexample to Vaught's conjecture is not \aleph_1 -categorical. The crux of Shelah's argument is an appeal to the non-definability of well-order in $L_{\omega_1, \omega}$.

The second step is to require that for each countable fragment Δ there are only countably many Δ -types over the empty set. If ϕ has arbitrarily large models this is easy by using Ehrenfeucht-Mostowski models. But if not, the only known argument is from few models in \aleph_1 and depends on a subtle argument of Keisler [Kei71] (See also Appendix C of [Bal09].)

Theorem 3.6 (Keisler) *If ϕ has $< 2^{\aleph_1}$ models of cardinality \aleph_1 , then each model of ϕ is Δ -small for every countable Δ .*

Now Theorem's 3.5 and 3.6 immediately yield.

Theorem 3.7 [Shelah] *If ϕ has $< 2^{\aleph_1}$ models of cardinality \aleph_1 , then there is a complete sentence ψ such that ψ implies ϕ and ψ has an uncountable model. In particular, if ϕ is \aleph_1 -categorical there is a Scott sentence for the model in \aleph_1 , i.e. the model in \aleph_1 is small. So an atomic class \mathbf{K} is associated with ϕ .*

It is easy to construct a sentence ϕ such that no completion has an uncountable model, i.e. no uncountable model is small. Let τ contain binary relations E_n for $n < \omega$. Let ϕ assert that the E_n are refining equivalence relations with binary splitting. And that there do not exist two distinct points that are E_n equivalent for all n . And add a countable set A of constants that realize a dense set of paths. Now every uncountable model realizes uncountably many distinct types over A .

We have the following question, which is open if $\kappa > \aleph_1$.

Question 3.8 *If ϕ is κ -categorical must the model in κ be small?*

Thus for technical work we will consider the class of atomic models of first order theories. Our notion of type will be the usual first order one - but we must define a restricted Stone space.

Definition 3.9 *Let A be an atomic set; $S_{\text{at}}(A)$ is the collection of $p \in S(A)$ such that if $\mathfrak{a} \in \mathbb{M}$ realizes p , $A\mathfrak{a}$ is atomic.*

Here \mathbb{M} is the monster model for the ambient theory T ; in interesting cases it is not atomic. And the existence⁵ of a monster model for the atomic class associated with a sentence categorical in some set of cardinals is a major project. (It follows from excellence. After Theorem 4.3, we see under VWGCH categoricity up to \aleph_ω is sufficient).

Definition 3.10 *\mathbf{K} is λ -stable if for every model M in \mathbf{K} (thus necessarily atomic) with cardinality λ , $|S_{\text{at}}(M)| = \lambda$.*

The insistence that M be a model is essential. The interesting examples of pseudo-exponential fields, covers of Abelian varieties and the basic examples of Marcus and Julia Knight all are ω -stable but have countable sets A with $|S_{\text{at}}(A)| > \aleph_0$.

With somewhat more difficulty than the first order case, one obtains:

Theorem 3.11 *For a class \mathbf{K} of atomic models, ω -stable implies stable in κ for all κ .*

A fundamental result in model theory is Morley's proof that an \aleph_1 -categorical first order theory is ω -stable. This argument depends on the compactness theorem in a number of ways. The key idea is to construct an Ehrenfeucht-Mostowski model over a well-order of cardinality \aleph_1 . Such a model realizes only countably many types over any countable submodel. But the existence of the model depends on a compactness argument in the proof of the Ehrenfeucht-Mostowski theorem. Further, this only contradicts ω -stability because amalgamation allows the construction from a model M_0 in \aleph_0 that has uncountably many types over it an elementary extension M_1 of M_0 with power \aleph_1 that realizes all of them. And again amalgamation in the first order case is a consequence of compactness. In $L_{\omega_1, \omega}$, the work of Keisler and Shelah evades the use of compactness – but at the cost of set theoretic hypotheses.

Theorem 3.12 (Keisler-Shelah) *Let \mathbf{K} be the atomic models of a countable first order theory. If \mathbf{K} is \aleph_1 -categorical and $2^{\aleph_0} < 2^{\aleph_1}$ then \mathbf{K} is ω -stable.*

This proof uses WCH directly and weak diamond via ‘The Only Completely General Result’. That is, from amalgamation failure of ω -stability yields a model of cardinality \aleph_1 that realizes uncountably many types from $S_{\text{at}}(M)$ for a countable model M . Naming the elements of M yields a theory which has uncountably many types over the empty set. Thus by Theorem 3.6 the new theory has 2^{\aleph_1} models in \aleph_1 and (since $2^{\aleph_0} < 2^{\aleph_1}$) so does the original theory.

Is CH is necessary? More precisely, does $\text{MA} + \neg \text{CH}$ imply there is a sentence of $L_{\omega_1, \omega}$ that is \aleph_1 -categorical but

⁵The difficulties we discuss here concern obtaining amalgamation. For simplicity, think only of gaining a monster model in λ with $\lambda^{<\lambda} = \lambda$. Weakening that hypothesis is a different project (See [Bal09, Hod93]) or any first order stability book for comments on the cardinality question.)

- a) is not ω -stable
- b) does not satisfy amalgamation even for countable models.

There is such an example in $L_{\omega_1, \omega}(Q)$ but Laskowski (unpublished) showed the example proposed for $L_{\omega_1, \omega}$ by Shelah[She87, She09a] fails. The previous question is a specific strategy for answering the next question.

Question 3.13 *Is categoricity in \aleph_1 of a sentence of $L_{\omega_1, \omega}$ absolute (with respect suitable forcings)?*

By suitable, I mean that, e.g., it is natural to demand cardinal preserving. This result has resisted a number of attempts although as we lay out in Section 5, many other fundamental notions of the model theory of $L_{\omega_1, \omega}$ are absolute.

4 The Conjecture for $L_{\omega_1, \omega}$

Using the notion of splitting, a nice theory of independence (Definition 5.6) can be defined for ω -stable atomic classes [She83a, She83b, Bal09]. This allows the formulation of the crucial notion of excellence and the proof of a version of Morley's theorem. We won't discuss the details but outline some aspects of the argument. These results are non-trivial but the exposition of the entire situation in [Bal09] occupies less than 100 pages.

The concept of an independent system of models is hard to grasp although it is playing an increasing role in many areas of model theory. Rather than repeating the notation heavy definition (see [She83b, Bal09, Les05] order first stability texts.) I give a simple example. Let X be a set of n algebraically independent elements in an algebraically closed field. For each $Y \subsetneq X$, let M_Y be the algebraic closure of Y . The M_Y form a independent system of $2^n - 1$ -models. This is exactly the concept needed in Zilber's theory of quasiminimal excellence. For Shelah's more general approach the notion is axiomatized using the independence notion from the previous paragraph. In the example, there is clearly a prime model over the union of the independent system. In various more complicated algebraic examples (e.g. [BZ00]) the existence of such a prime model is non-trivial. Here we discuss how to find one from model theoretic hypotheses.

Definition 4.1 *Let K be an atomic class. K is excellent if K is ω -stable and any of the following equivalent conditions hold.*

For any finite independent system of countable models with union C :

1. $S_{at}(C)$ is countable.
2. There is a unique primary model over C .
3. The isolated types are dense in $S_{at}(C)$.

The key point is that this is a condition of ‘ n -dimensional amalgamation’. A primary model is a particularly strong way of choosing a prime model over C . Thus, condition ii) specifies the existence of a strong kind of amalgamation of n independent models. This definition emphasizes the contrast of the current situation with first order logic; condition 1) does *not* follow from ω -stability. See [Bal09] for details of the notation.

Note that excellence is a condition on countable models. It has the following consequence for models in *all* cardinalities. The key to this extension is the proof that n -dimensional amalgamation in \aleph_n implies $n - 1$ -dimensional amalgamation in \aleph_{n+1} . Thus amalgamation for all n in \aleph_0 implies amalgamation for all n below \aleph_ω and then for all cardinals by a short argument.

Theorem 4.2 (Shelah (ZFC)) *If an atomic class \mathbf{K} is excellent and has an uncountable model then*

1. \mathbf{K} has models of arbitrarily large cardinality;
2. *Categoricity in one uncountable power implies categoricity⁶ in all uncountable powers.*

This result is in ZFC but extensions of set theory are used to obtain excellence. Recall that by VWGCH we mean the assertion: $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for $n < \omega$. The following is an immediate corollary of Theorem 4.6.

Theorem 4.3 (Shelah (VWGCH)) *An atomic class \mathbf{K} that is categorical in \aleph_n for each $n < \omega$ is excellent.*

We remarked after Definition 3.9 on the difficulty of constructing a monster model for an atomic class associated with a sentence categorical in some power. Of course such a monster model in appropriate cardinalities is immediate from the amalgamation property. But, even assuming categoricity up to \aleph_ω , we need to use the VWGCH to get excellence, then derive amalgamation and finally a monster model.

The requirement of categoricity below \aleph_ω in Theorem 4.3 is essential. Baldwin-Kolesnikov [BK09] (refining [HS90]) show:

Theorem 4.4 *For each $2 \leq k < \omega$ there is an $L_{\omega_1, \omega}$ -sentence ϕ_k such that:*

1. ϕ_k has an atomic model in every cardinal.
2. ϕ_k is categorical in μ if $\mu \leq \aleph_{k-2}$;
3. ϕ_k is not categorical in any μ with $\mu > \aleph_{k-2}$;
4. ϕ_k has the (disjoint) amalgamation property;

Note that of course the ϕ_k are not excellent. There is one further refinement on the ‘wide’ vrs ‘deep’ metaphor. How wide?

⁶In contrast to some authors, we say \mathbf{K} is categorical in κ if there is *exactly* one model in cardinality κ .

Definition 4.5 *We say*

1. \mathbf{K} has few models in power λ if $I(\mathbf{K}, \lambda) < 2^\lambda$.
2. \mathbf{K} has very few models in power \aleph_n if $I(\mathbf{K}, \aleph_n) \leq 2^{\aleph_{n-1}}$.

These are equivalent under GCH. And Shelah argues on the last couple of pages of [She83b] (see also [She0x]) that they are equivalent under $\neg 0^+$. But in general we have a theorem and a conjecture [She83a, She83b], which differ only in the word ‘very’.

Theorem 4.6 (Shelah) *(For $n < \omega$, $2^{\aleph_n} < 2^{\aleph_{n+1}}$.) An atomic class \mathbf{K} that has at least one uncountable model and that has very few models in \aleph_n for each $n < \omega$ is excellent.*

Conjecture 4.7 (Shelah) *(For $n < \omega$, $2^{\aleph_n} < 2^{\aleph_{n+1}}$.) An atomic class \mathbf{K} that has at least one uncountable model and that has few models in \aleph_n for each $n < \omega$ is excellent.*

The proof of Theorem 4.6 uses the technology of atomic classes very heavily. But the calculation of the categoricity spectrum in Theorem 4.2.2 can be lifted to arbitrary sentences of $L_{\omega_1, \omega}$ by a calculation [She83a, She83b], reported as Theorem 25.19 of [Bal09].

5 Absoluteness of properties of atomic classes

As remarked in the introduction, one of the significant attributes of first order stability theory is that the basic notions: stable, ω -stable, superstable, \aleph_1 -categoricity can be seen absolute in very strong ways. We sketch proofs of similar results, except the open \aleph_1 -categoricity, for $L_{\omega_1, \omega}$. This section and the appendix tie together some results which are folklore with the use of well-known methods which are systematically applied to discuss the case of $L_{\omega_1, \omega}$. We are indebted for discussions with Alf Dolich, Paul Larson, Chris Laskowski, and Dave Marker for clarifying the issues. One of the few places model theoretic absoluteness issues have been addressed in print is [She09b].

For example a first order theory T is unstable just if there is a formula $\phi(\mathbf{x}, \mathbf{y})$ such for every n

$$T \models (\exists \mathbf{x}_1, \dots, \mathbf{x}_n \exists \mathbf{y}_1, \dots, \mathbf{y}_n) \bigwedge_{i < j} \phi(\mathbf{x}_i, \mathbf{y}_j) \wedge \bigwedge_{i \geq j} \neg \phi(\mathbf{x}_i, \mathbf{y}_j)$$

This is an arithmetic statement and so is absolute by basic properties of absoluteness [Kun80, Jec87]. In first order logic, ω -stability is Π_1^1 ; there is no consistent tree⁷ $\{\phi_i^{\sigma(i)}(x_\sigma, \mathbf{a}_\sigma \upharpoonright n) : \sigma \in 2^\omega, i < \omega\}$. With a heavier use of effective descriptive set theory, suggested by Dave Marker, the same applies for the atomic class case.

To demonstrate absoluteness of various concepts of infinitary logic we need the full strength of the Shoenfield absoluteness lemma. In this section, we work with *atomic*

⁷We use the convention that $\phi^{\sigma(i)}\phi(x)$ denotes $\phi(x)$ or $\neg\phi(x)$ depending on whether $\sigma(i)$ is 0 or 1.

classes, Definition 3.2. We noted Shelah's observation Theorem 3.7 that each \aleph_1 -categorical sentence of $L_{\omega_1, \omega}$ determines such a class. In this section we first show absoluteness for various properties of atomic classes. In the last theorem, we show that the properties for sentences of $L_{\omega_1, \omega}$ remain absolute although in some cases they are more complex. The Appendix (written by David Marker) makes a precise definition of a formula in $L_{\omega_1, \omega}$ as a subset of $\omega^{<\omega}$ so that we can apply descriptive set theoretic techniques. It gives an effective analysis of the transformation in Theorem 3.3. This sets notation for the rest of the paper and leads to the proof for example that the collection of complete sentences is complete Π_1^1 .

Theorem 5.1 (Shoenfield absoluteness Lemma) *If*

1. $V \subset V'$ are models of ZF with the same ordinals and
2. ϕ is a lightface Π_2^1 predicate of a set of natural numbers

then for any $A \subset \mathbb{N}$, $V \models \phi(A)$ iff $V' \models \phi(A)$.

Note that this trivially gives the same absoluteness results for Σ_2^1 -predicates.

Lemma 5.2 (Atomic models) 1. ' T has an atomic model' is an arithmetic property of T .

2. ' M is an atomic model of T ' is an arithmetic property of M and T .
3. For any vocabulary τ , the class of countable atomic τ -structures, M , is Borel.

Proof. The first condition is given by: for every formula $\phi(\mathbf{x})$ there is a $\psi(\mathbf{x})$, consistent with T , such that $\psi(\mathbf{x}) \rightarrow \phi(\mathbf{x})$ and for every $\chi(\mathbf{x})$, $\psi(\mathbf{x}) \rightarrow \chi(\mathbf{x})$ or $\psi(\mathbf{x}) \rightarrow \neg\chi(\mathbf{x})$. Let $\theta(M, T)$ be the arithmetic predicate of the reals M, T asserting that T is the theory of M . Now the second condition is a Δ_1^1 -predicate of M given by: there exists (for all) T such that $\theta(M, T)$, for every $\mathbf{a} \in M$ there exists a T -atom $\psi(\mathbf{x})$ such that $M \models \psi(\mathbf{a})$. $\square_{5.2}$

Earlier versions of this paper had weaker characterizations (e.g. a Σ_2^1 characterization of ω -stability). Marker pointed out the application of Harrison's theorem, Fact 5.4.ii, to improve the result to Π_1^1 .

Definition 5.3 $x \in \omega^\omega$ is hyperarithmetical if $x \in \Delta_1^1$. x is hyperarithmetical in y , written $x \leq_{\text{hyp}} y$, if $x \in \Delta_1^1(y)$.

Fact 5.4 1. The predicate $\{(x, y) : x \leq_{\text{hyp}} y\}$ is Π_1^1 .

2. If $K \subset \omega^\omega$ is Σ_1^1 , then for any y , K contains an element which is not hyperarithmetical in y if and only if K contains a perfect set.

The unrelativized version of statement 1) is II.1.4.ii of [Sac90]; the relativized version is 7.15 of [Mar]. Again, the unrelativized version of statement 2) is III.6.2 of [Sac90]; in this case the relativization is routine. $\square_{5.4}$

In the next theorem, the atomic set A must be regarded as element of ω^ω . There are at least two ways to think of this: 1) a pair (M, A) where M is a countable model of T and A a subset (automatically atomic or 2) as a pair (A, Φ) where Φ is the diagram of A as submodel of the monster model \mathbb{M} .

Lemma 5.5 (Marker) *Let K be an atomic class (Definition 3.2) with a countable complete first order theory T .*

1. *Let A be a countable atomic set. The predicate of p and A , ‘ p is in $S_{\text{at}}(A)$ ’, is arithmetic.*
2. *‘ $S_{\text{at}}(A)$ is countable’ is a Π_1^1 -predicate of A .*

Proof. i) Note first that ‘ $q(\mathbf{x})$ is a principal type over \emptyset in T ’ is an arithmetic property. Now p is in $S_{\text{at}}(A)$ if and only if for all $\mathbf{a} \in A$, $p \upharpoonright \mathbf{a}$ is a principal type. So this is also arithmetic.

ii) By i), the set of p such that ‘ p is in $S_{\text{at}}(A)$ ’ is arithmetic (*a fortiori* Σ_1^1) in A , so by Lemma 5.4.ii, each such p is hyperarithmetic in A . Since the continuum hypothesis holds for Σ_1^1 -sets, ‘ $S_{\text{at}}(A)$ is countable’ is formalized by:

$$(\forall p)[p \in S_{\text{at}}(A) \rightarrow (p \leq_{\text{hyp}} A)],$$

which is Π_1^1 .

□_{5.5}

In order to show the absoluteness of excellence we need some more detail on the notion of independence. We will use item i) of Definition 4.1. The independent families of models [She83b, Bal09] in that definition are indexed by subsets of n with strictly less than n elements; we denote this partial order by $\mathcal{P}^-(n)$. We will show that independence of models is an arithmetic property.

Definition 5.6 1. *A complete type p over A splits over $B \subset A$ if there are $\mathbf{b}, \mathbf{c} \in A$ which realize the same type over B and a formula $\phi(\mathbf{x}, \mathbf{y})$ with $\phi(\mathbf{x}, \mathbf{b}) \in p$ and $\neg\phi(\mathbf{x}, \mathbf{c}) \in p$.*

2. *Let ABC be atomic. We write $A \underset{C}{\perp} B$ and say A is free or independent from B over C if for any finite sequence \mathbf{a} from A , $\text{tp}(\mathbf{a}/B)$ does not split over some finite subset of C .*

Lemma 5.7 *Let T be a complete countable first order theory. The properties that the class of atomic models of T is*

1. *ω -stable*
2. *excellent*

are each given by a Π_1^1 formula of set theory and so are absolute.

Proof. 1) The class of atomic models of T is ω -stable if and only if for every atomic A , ‘ $S_{\text{at}}(A)$ is countable’. This property is Π_1^1 by Lemma 5.5.

2) The class of atomic models of T is excellent if and only if for any finite set of countable atomic models $\{A_s : s \in \mathcal{P}^-(n)\}$ that form an independent system, with $A = \bigcup\{A_s : s \in \mathcal{P}^-(n)\}$, $S_{\text{at}}(A)$ is countable. Here we have universal quantifiers over finite sequences of models (using a pairing function, this is quantifying over a single real). The stipulation that the diagram is independent requires repeated use of

the statement $A \underset{C}{\downarrow} B$, where A, B, C are finite unions of the models in the independent system. This requires quantification over finite sequences from the A_s ; thus, it is arithmetic. The assertion ‘ $S_{\text{at}}(A)$ is countable’ is again π_1^1 by Lemma 5.5 and we finish. $\square_{5.7}$

Lemma 5.8 *The property that an atomic class \mathbf{K} has arbitrarily large models is absolute. In fact it is Σ_1^1 .*

Proof. Let \mathbf{K} be the class of atomic models of a first order theory T in a vocabulary τ . \mathbf{K} has arbitrarily large models if and only there are \hat{T} , $\hat{\tau}$, M and C such that \hat{T} is a Skolemization of T in a vocabulary $\hat{\tau}$ and M is a countable model of \hat{T} such that $M \upharpoonright \tau$ is atomic and M contains an infinite set C of $\hat{\tau}$ -indiscernibles. This formula is Σ_1^1 . $\square_{8.8}$

Finally, following Lessmann [Les05, Bal09], we prove that the absolute ‘Baldwin-Lachlan’-characterization of first order \aleph_1 -categoricity has a natural translation to the $L_{\omega_1, \omega}$ situation; the resulting property of atomic classes is absolute and in ZFC it implies \aleph_1 -categoricity. But we do not see how to derive it from \aleph_1 -categoricity without using the Continuum hypothesis. We need some definitions. To be a bit more specific we speak of Vaughtian triples instead of Vaughtian pairs.

Definition 5.9 *The formula $\phi(x, \mathbf{c})$ with $\mathbf{c} \in M \in \mathbf{K}$, is big if for any $M' \supseteq A$ with $M' \in \mathbf{K}$ there exists an N' with $M' \prec_{\mathbf{K}} N'$ and with a realization of $\phi(x, \mathbf{c})$ in $N' - M'$.*

This definition has no requirements on the cardinality of M, M', N' so it is saying that $\phi(x, \mathbf{c})$ has as many solutions as the size of the largest models in \mathbf{K} . This condition is equivalent to one on countable models. A translation of Lemma 25.2 of [Bal09] gives:

Lemma 5.10 *Let $A \subseteq M$ and $\phi(x, \mathbf{c})$ be over A . The following are equivalent.*

1. *There is an N with $M \prec N$ and $c \in N - M$ satisfying $\phi(x, \mathbf{c})$;*
2. *$\phi(x, \mathbf{c})$ is big.*

The significance of this remark is that it makes ‘ $\phi(x, \mathbf{c})$ is big’ a Σ_1^1 predicate.

Definition 5.11 *1. A triple (M, N, ϕ) where $M \prec N \in \mathbf{K}$ with $M \neq N$, ϕ defined over M , ϕ big, and $\phi(M) = \phi(N)$ is called a Vaughtian triple.*

2. *We say \mathbf{K} admits (κ, λ) , witnessed by ϕ , if there is a model $N \in \mathbf{K}$ with $|N| = \kappa$ and $|\phi(N)| = \lambda$ and ϕ is big.*

Now we have the partial characterization.

Lemma 5.12 *Let \mathbf{K} be a class of atomic models. If \mathbf{K} is ω -stable and has no Vaughtian triples then \mathbf{K} is \aleph_1 -categorical. The hypothesis of this statement is Σ_2^1 .*

Proof. The sufficiency of the condition is found by tracing results in [Bal09]: ω -stability gives the existence of a quasiminimal formula ϕ . Note from the proof of Theorem 24.1 in [Bal09] that ω -stability is sufficient to show that there are prime models over independent subsets of cardinality \aleph_1 . (The point of excellence is that higher dimensional amalgamation is needed to extend this result to larger sets.) So if $|M| = \aleph_1$, there is an $N \prec_{\mathbf{K}} M$ which is prime over a basis for $\phi(M)$. As noted in Chapter 2 of [Bal09], this determines N up to isomorphism (again without use of excellence because we are in \aleph_1). So we are done unless $N \not\approx M$. But then Löwenheim-Skolem gives us a countable Vaughtian triple, contrary to hypothesis. $\square_{5.12}$

Consequence 5.13 *Let \mathbf{K} be a class of atomic models. \aleph_1 -categoricity of \mathbf{K} is absolute between models of set theory that satisfy either of the following conditions.*

1. \mathbf{K} has arbitrarily large members and \mathbf{K} has amalgamation in \aleph_0 , or
2. $2^{\aleph_0} < 2^{\aleph_1}$.

Proof. Each hypothesis implies the characterization in Lemma 5.12. $\square_{5.13}$

Note, the hypothesis of condition 1) is absolute. It seems unlikely that \aleph_1 -categoricity implies the existence of arbitrarily large models in \mathbf{K} ; but no counterexample has yet been constructed. The use of the continuum hypothesis is central to current proofs that \aleph_1 -categoricity implies amalgamation and ω -stability. But [FK0x] have shown (employing standard forcings) that for each AEC \mathbf{K} that fails amalgamation in \aleph_0 , there is a model of set theory such that in that model $2^{\aleph_0} = 2^{\aleph_1}$, \mathbf{K} continues to fail amalgamation in \aleph_0 , and \mathbf{K} has 2^{\aleph_1} models in \aleph_1 .

Theorem 5.14 *Each of the properties that a complete sentence of $L_{\omega_1, \omega}$ is ω -stable, excellent, or has no two-cardinal models is Σ_2^1 .*

Proof. Let $Q(T)$ denote any of the conditions above as a property of the first order theory T in a vocabulary τ^* . Now write the following properties of the complete sentence ϕ in vocabulary τ .

1. ϕ is a complete sentence.
2. There exists a $\tau^* \supseteq \tau$ and τ^* theory T satisfying the following.
 - (a) T is a complete theory that has an atomic model.
 - (b) The reduct to τ of any atomic model of T satisfies ϕ .
 - (c) There is a model M of ϕ and there exists an expansion of M to an atomic model of T .
 - (d) $Q(T)$.

Proof. Note that the most complicated conditions (including the universal quantifier in 2 and further ones in particular clause) 2b) and 2d) are Σ_2^1 . $\square_{5.14}$

6 Complexity

We prove the following claim. This result was developed in conversation with Martin Koerwien and Sy Friedman at the CRM Barcelona and benefitted from further discussion with Dave Marker.

Claim 6.1 *The class of countable models whose automorphism groups admit a complete left invariant metric is Π_1^1 but not Σ_1^1 .*

Our proof is by propositional logic from known results of Gao[Gao96] and Deissler [Dei77].

Definition 6.2 *A countable model is minimal (equivalently non-extendible) if it has no proper $L_{\omega_1, \omega}$ -elementary submodel.*

We showed in Lemma 5.2 that the class of atomic structures is Borel. The following claim is an easy back and forth.

Claim 6.3 *If M is atomic, τ -elementary submodel is the same as $L_{\omega_1, \omega}(\tau)$ -elementary submodel.*

Claim 6.3 shows an atomic model is minimal iff it is minimal in first order logic. Note that the class of minimal models is obviously Π_1^1 . Now if the class of minimal models were Borel, it would follow that the class of minimal atomic (equal first order minimal prime) models is also Borel. But Corollary 2.6 of Deissler [Dei77] asserts for first order theories:

Lemma 6.4 (Deissler) *There is a countable vocabulary τ such that the class of minimal prime models for τ is not Σ_1^1 .*

Gao characterized non-extendible models in terms of metrics on their automorphism group.

Lemma 6.5 (Gao) *The following are equivalent:*

1. *$\text{Aut}(M)$ admits a compatible left-invariant complete metric.*
2. *There is no $L_{\omega_1, \omega}$ -elementary embedding from M into itself which is not onto.*

So we can transfer to the characterization of automorphism groups and prove Claim 6.1.

Malicki recently proved a related result: the class of Polish groups with a complete left invariant metric) is Π_1^1 but not Σ_1^1 .

7 Conclusion

The spectrum problem for first order theories motivated many technical developments that eventually had significant algebraic consequences. A similar possibility for application of infinitary logic to algebraic problems is suggested by Zilber's program [Zil06, Zil04]. But the basic development is far more difficult and less advanced. The notion of excellence provides one useful context. And others are being developed under the guise of abstract elementary classes and metric abstract elementary classes. But while first order stability theory is developed in ZFC, the current development of the model theory of $L_{\omega_1, \omega}$ uses a (rather weak) extension of set theory: the VWGCH. This raises both model theoretic and set theoretic questions. The proof of the 'one completely general result', Theorem 2.4, is a fundamentally combinatorial argument using no sophisticated model theoretic lemmas. The current proof uses $2^\lambda < 2^{\lambda^+}$. Can this hypothesis be removed?

Like first order logic such fundamental definitions of $L_{\omega_1, \omega}$ as satisfaction, ω -stability, and excellence are absolute. And in fact the complexity of their description can often be computed. But while \aleph_1 -categoricity is seen (by a model theoretic argument) to be absolute in the first order case, this issue remains open for $L_{\omega_1, \omega}$.

We have also investigated the complexity of various properties of $L_{\omega_1, \omega}$ -sentences and associated atomic classes. It is shown in Lemma 8.7 that the graph of the translation from a sentence to a finite diagram (T, Γ) is arithmetic. In Theorem 5.14, we avoided a precise calculation of the translation from a complete sentence to the atomic models of a first order theory. The tools of the appendix should allow a careful computation of this complexity. Note that while, for example, we showed ω -stability was Π_1^1 as a property of an atomic class, we only showed it to be Σ_2^1 as a property of the $L_{\omega_1, \omega}$ -sentence.

8 Appendix: Basic definability notions for $L_{\omega_1, \omega}$ by David Marker

Fix a vocabulary τ and let \mathbb{X}_τ be the Polish space of countable τ -structures with universe ω . Our first goal is to describe the collection of codes for $L_{\omega_1, \omega}(\tau)$ -formulas. This is analogous to the construction of Borel codes in descriptive set theory.

Definition 8.1 1. A labeled tree is a non-empty tree $T \subseteq \omega^{<\omega}$ with functions l and v with domain T such that for any $\sigma \in T$ one of the following holds:

- σ is a terminal node of T then $l(\sigma) = \psi$ where ψ is an atomic τ -formula and $v(\sigma)$ is the set of free variables in ψ ;
- $l(\sigma) = \neg$, $\sigma \hat{\ } 0$ is the unique successor of σ in T and $v(\sigma) = v(\sigma \hat{\ } 0)$;
- $l(\sigma) = \exists v_i$, $\sigma \hat{\ } 0$ is the unique successor of σ in T and $v(\sigma) = v(\sigma \hat{\ } 0) \setminus \{i\}$;
- $l(\sigma) = \bigwedge$ and $v(\sigma) = \bigcup_{\sigma \hat{\ } i \in T} v(\sigma \hat{\ } i)$ is finite.

2. A formula ϕ is a well founded labeled tree (T, l, v) . A sentence is a formula where $v(\emptyset) = \emptyset$.

Proposition 8.2 *The set of labeled trees is arithmetic. The set of formulas is Π_1^1 , as is the set of sentences.*

Now it is easy to see:

Proposition 8.3 *There is $R(x, y) \in \Pi_1^1$ and $S(x, y) \in \Sigma_1^1$ such that if ϕ is a sentence and $M \in \mathbb{X}_\tau$, then*

$$M \models \phi \Leftrightarrow R(M, \phi) \Leftrightarrow S(M, \phi).$$

In particular, $\{(M, \phi) : \phi \text{ is a sentence and } M \models \phi\}$ is Π_1^1 , but for any fixed ϕ , $\text{Mod}(\phi) = \{M \in \mathbb{X}_\tau : M \models \phi\}$ is Borel, indeed $\Delta_1^1(\phi)$.

Proof. We define a predicate ‘ f is a truth definition for the labeled tree (T, l, v) in M ’ as follows.

- The domain of f is pairs $(\sigma, M\mu)$ where $\sigma \in T$ and $\mu : v(\sigma) \rightarrow M$ is an assignment of the free variables at node σ and $f(\sigma, \mu) \in \{0, 1\}$.
- If $l(\sigma) = \psi$ an atomic formula, then $f(\sigma, \mu) = 1$ if and only if ψ is true in M when we use μ to assign the free variables.
- If $l(\sigma) = \neg$, then $f(\sigma, \mu) = 1$ if and only if $f(\sigma^{\wedge}0, \mu) = 0$.
- If $l(\sigma) = \exists v_i$ there are two cases. If $v_i \in v(\sigma^{\wedge}0)$, then $f(\sigma, \mu) = 1$ if and only if there is $a \in M$ such that $f(\sigma^{\wedge}0, \mu^*) = 1$, where $\mu^* \supset \mu$ is the assignment where $\mu^*(v_i) = a$. Otherwise, $f(\sigma, \mu) = f(\sigma^{\wedge}0, \mu)$.
- If $l(\sigma) = \bigwedge$, then $f(\sigma, \mu) = 1$ if and only if $f(\sigma^{\wedge}i, \mu|v)(\sigma^{\wedge}i) = 1$ for all i such that $\sigma^{\wedge}i \in T$.

This predicate is arithmetic. If ϕ is a sentence, there is a unique truth definition f for ϕ in M . Let

$R(x, y) \Leftrightarrow x \in \mathbb{X}_\tau$ and y is a labeled tree and $f(\emptyset, \emptyset) = 1$ for all truth definitions f for y in x
and

$S(x, y) \Leftrightarrow y$ is a labeled tree and there is a truth definition f for y in x such that $f(\emptyset, \emptyset) = 1$. □_{8.3}

Notation 8.4 *We write that a property of a set of reals is $\Pi_1^1 \cap \Sigma_1^1$ if it is defined by the conjunction of a Π_1^1 and a Σ_1^1 formula.*

Proposition 8.5 $\{\phi : \phi \text{ is a satisfiable sentence}\}$ is $\Pi_1^1 \cap \Sigma_1^1$, but neither Π_1^1 nor Σ_1^1 .

Proof. ‘ ϕ is a sentence’ is Π_1^1 ; ‘there is a model for ϕ ’ is equivalent to $\exists x S(x, \phi)$ which is Σ_1^1 .

The set of satisfiable sentences is not Σ_1^1 since otherwise the set of underlying trees would be a Σ_1^1 -set of trees and there would be a countable bound (e.g. Theorem 3.2 of [MW85]), on their heights.

We show that the set of satisfiable sentences is not Π_1^1 by constructing a reduction of non-well ordered linear orders to satisfiable sentences.

Let $\tau = \{U, V, <, s, f, 0, c_n : n \in \omega\}$

For each linear order \prec of ω we write down an $L_{\omega_1, \omega}$ sentence ϕ_\prec asserting:

- the universe is the disjoint union of U and V ;

- $U = \{c_0, c_1, \dots\}$ all of which are distinct;
- $<$ is a linear order of U ;
- $c_n < c_m$, if $n \prec m$;
- s is a successor function on V and $V = \{0, s(0), s(s(0)), \dots\}$;
- $f : V \rightarrow U$ and $f(s(n)) < f(n)$ for all n .

It \prec is not a well order, and $n_0 \succ n_1 \succ \dots$ is an infinite descending chain, then by defining $f(n) = c_n$ we get a model of ϕ_{\prec} . On the other hand if \prec is a well order we can find no model of ϕ_{\prec} .

Thus $\prec \mapsto \phi_{\prec}$ is a reduction of non-well-ordered linear orders to $\{\phi : \phi \text{ is satisfiable}\}$ which is impossible if satisfiability is Π_1^1 . $\square_{8.5}$

We now effectivize Chang's observation (Lemma 3.3) that for each sentence ϕ in $L_{\omega_1, \omega}$ we can find a first order theory T^* in a vocabulary τ^* and a countable set Γ of partial τ^* -types such that the models of ϕ are exactly the τ -reducts of models of T^* that omit all the types in Γ .

Definition 8.6 A Chang-assignment to a labeled tree (T, l, v) is a pair of functions S, γ with domain T such that $S(\sigma)$ is a set of sentences in the vocabulary $\tau_\sigma = \tau \cup \{R_\tau : \tau \supseteq \sigma\}$, where R_τ is a relation symbol in $|v(\tau)|$ -variables and $\gamma(\sigma)$ is a function with domain ω such that each $\gamma(\sigma)(n)$ is a partial τ_σ type.⁸ We also require:

- if $l(\sigma) = \psi$ is atomic, $S(\sigma) = \{\forall \bar{v}(\sigma)(R_\sigma(\bar{v}) \leftrightarrow \psi)\}$, and each $\gamma(\sigma)(i) = \{v_1 \neq v_1\}$;
 - if $l(\sigma) = \neg$, then $S(\sigma) = S(\sigma \hat{\ } 0) \cup \{\forall \bar{v}(\sigma)R_\sigma \leftrightarrow \text{Neg}R_{\sigma \hat{\ } 0}\}$ and $\gamma(\sigma) = \gamma(\sigma \hat{\ } 0)$.
 - if $l(\sigma) = \exists v_i$, then $S(\sigma) = S(\sigma \hat{\ } 0) \cup \{\forall \bar{v}(\sigma)R_\sigma \leftrightarrow \exists v_i R_{\sigma \hat{\ } 0}\}$ and $\gamma(\sigma) = \gamma(\sigma \hat{\ } 0)$.
 - if $l(\sigma) = \bigwedge$; then $S(\sigma) = \bigcup_{\sigma \hat{\ } i \in T} S(\sigma \hat{\ } i) \cup \{\forall \bar{v}(\sigma)(R_\sigma \rightarrow R_{\sigma \hat{\ } i}) : \sigma \hat{\ } i \in T\}$.
- Fix $\mu : \omega \times \omega \rightarrow \omega$ be a pairing function. Let

$$\gamma(\sigma)(0) = \{R_\sigma, \neg R_{\sigma \hat{\ } i} : \sigma \hat{\ } i \in T\}$$

and

$$\gamma(\sigma)(\mu(i, n) + 1) = \begin{cases} \gamma(\sigma \hat{\ } i)(n) & \text{if } \sigma \hat{\ } i \in T \\ \{v_1 \neq v_1\} & \text{otherwise.} \end{cases}$$

In other words $\gamma(\sigma)$ lists all the types listed by the successors of σ and the additional type $\{R_\sigma, \neg R_{\sigma \hat{\ } i} : \sigma \hat{\ } i \in T\}$.

It is now easy to see:

Lemma 8.7 The predicate “ (S, γ) is a Chang-assignment for the labeled tree (T, l, v) ” is arithmetic. If ϕ is a sentence then there is a unique Chang-assignment for ϕ .

⁸We allow relation symbols in 0 variables, but these could easily be eliminated.

To simplify notation we will call (T, Γ) the Chang-assignment, where T is the theory $S(\emptyset)$ and Γ is the set of types $\gamma(\emptyset)(0), \gamma(\emptyset)(1), \dots$

The following remark is implicit in [GS86].

Lemma 8.8 *The property that a sentence ϕ of $L_{\omega_1, \omega}$ has arbitrarily large models is absolute. In fact it is $\Pi_1^1 \cap \Sigma_1^1$, but neither Π_1^1 nor Σ_1^1 .*

Proof. A τ -sentence ϕ has arbitrarily large models if and only if there is a Chang-assignment (T, Γ) , $\tau^* \supseteq \tau$ and $T^* \supseteq T$ a Skolemized τ^* -theory such that there is a model of T^* omitting all types in Γ and containing an infinite set of τ^* -indiscernibles. This condition is Σ_1^1 once we restrict to the Π_1^1 -set of sentences.

For any sentence ϕ let ϕ^* be the sentence which asserts we have two sorts, the first of which is a model of ϕ and the second is an infinite set with no structure. Then ϕ is satisfiable if and only if ϕ^* has arbitrarily large models. Thus $\phi \mapsto \phi^*$ is a reduction of satisfiable sentences to sentences with arbitrarily large models. By Proposition 8.5, the set of sentences with arbitrarily large models is neither Σ_1^1 nor Π_1^1 . $\square_{8.8}$

Recall that an $L_{\omega_1, \omega}$ -sentence is *complete* if and only if it is satisfiable and any two countable models are isomorphic. This is easily seen to be Π_2^1 . Drawing on some results of Nadel, we show that in fact:

Theorem 8.9 $\{\phi : \phi \text{ is a complete sentence}\}$ is complete- Π_1^1 .

The argument requires some preparation. We begin by recalling the usual Karp-Scott back-and-forth analysis.

Definition 8.10 *If M and N are τ -structures, we inductively define \sim_α by:*

$(M, \mathbf{a}) \sim_0 (N, \mathbf{b})$ if $M \models \phi(\mathbf{a})$ if and only if $N \models \phi(\mathbf{b})$ for all atomic τ -formulas ϕ .

For all ordinals α , $(M, \mathbf{a}) \sim_{\alpha+1} (N, \mathbf{b})$ if for all $c \in M$ there is $d \in N$ such that $(M, \mathbf{a}, c) \sim_\alpha (N, \mathbf{b}, d)$ and for all $d \in N$ there is $c \in M$ such that $(M, \mathbf{a}, c) \sim_\alpha (N, \mathbf{b}, d)$

For all limit ordinals β , $(M, \mathbf{a}) \sim_\beta (N, \mathbf{b})$ if and only if $(M, \mathbf{a}) \sim_\alpha (N, \mathbf{b})$ for all $\alpha < \beta$.

A classical fact is that $(M, \mathbf{a}) \sim_\alpha (N, \mathbf{b})$ if and only if $M \models \phi(\mathbf{a}) \Leftrightarrow N \models \phi(\mathbf{b})$ for all formulas ϕ of quantifier rank at most α .

We say that ϕ has *Scott rank* α if α is the least ordinal such that if $M, N \models \phi$ and $(M, \mathbf{a}) \sim_\alpha (N, \mathbf{b})$ then $(M, \mathbf{a}) \sim_\beta (N, \mathbf{b})$ for all ordinals β .

We need to analyze the complexity of \sim_α .

Definition 8.11 *Let WO^* (the class of pseudo-well-orders) be the set of all linear orders R with domain ω such that:*

i) 0 is the R -least element;

ii) if n is not R -maximal, then there is y such that xRy and there is no z such that xRz and zRx , we say y is the R -successor of x and write $y = s_R(x)$. If $n \neq 0$ is not an R -successor we say it is an R -limit.

Note that WO^* , $s_R(n) = m$ and ‘ n is an R -limit’ are arithmetic.

Definition 8.12 We say that z is an R -analysis of M and N if

- i) $z \subseteq \omega \times \bigcup_{n \in \omega} (\omega^n \times \omega^n)$;
- ii) $(0, \mathbf{a}, \mathbf{b}) \in z$ if and only if $M \models \phi(\mathbf{a}) \leftrightarrow N \models \phi(\mathbf{b})$ for all quantifier free ϕ ;
- iii) if $(n, \mathbf{a}, \mathbf{b})$ and mRn , then $(m, \mathbf{a}, \mathbf{b})$;
- iv) $(s_R(n), \mathbf{a}, \mathbf{b}) \in z$ if and only if for all $c \in \omega$ there is $d \in \omega$ such that $(n, \mathbf{a} \hat{\ } c, \mathbf{b} \hat{\ } d) \in z$ and for all $d \in \omega$ there is $c \in \omega$ such that $(n, \mathbf{a} \hat{\ } c, \mathbf{b} \hat{\ } d) \in z$;
- v) if n is an R -limit, then $(n, \mathbf{a}, \mathbf{b}) \in z$ if and only if $(m, \mathbf{a}, \mathbf{b}) \in z$ for all mRn .

Note:

- ‘ $\{(z, R, M, N) : z \text{ is an } R\text{-analysis}\}$ ’ is arithmetic.
- Suppose R is a well-order of order type α . Let $\beta(n) < \alpha$ be the order type of $\{m : mRn\}$. If z is an R -analysis of M, N , then

$$(n, \mathbf{a}, \mathbf{b}) \in z \text{ if and only if } (M, \mathbf{a}) \sim_{\beta(n)} (N, \mathbf{b}).$$

In particular, there is a unique R -analysis of M, N .

We need two results of Mark Nadel.

Theorem 8.13 (Nadel) a) If ϕ is complete, then there is $M \models \phi$ with $M \leq_{\text{hyp}} \phi$.

b) If ϕ is complete then the Scott rank of ϕ is at most $\text{qr}(\phi) + \omega$ where $\text{qr}(\phi)$ is the quantifier rank of ϕ .

a) is [Nad74b] Theorem 2, while b) is [Nad74a] Theorem 5.1. For completeness we sketch the proofs.

a) Add new constants c_1, c_2, \dots to τ . Let F be a countable fragment such that $\phi \in F$, we can choose F arithmetic in ϕ . Let $S = \{s : s \text{ a finite set of } F\text{-sentences using only finitely many } c_i \text{ such that } \phi \models \exists \bar{v} \bigwedge_{\psi \in s} \psi(\bar{v})\}$. S is a consistency property. Since ϕ is complete,

$$\phi \models \theta \Leftrightarrow \forall M (M \models \phi \rightarrow M \models \theta) \Leftrightarrow \exists M (M \models \phi \wedge M \models \theta).$$

It follows that S is $\Delta_1^1(\phi, F)$ and hence $S \leq_{\text{hyp}} \phi$. Using the consistency property S one can easily construct $M \models \phi$ with $M \leq_{\text{hyp}} \phi$.

b) Let F be as above. Since ϕ is complete, there are only countably many F -types. By the Omitting Types Theorem for $L_{\omega_1, \omega}$, there is a model of ϕ where every element satisfies an F -complete formula. Since ϕ is complete, this is true in the unique countable model M .

The usual arguments show that we can do a back and forth in M with F -types. Thus if \mathbf{a}, \mathbf{b} in M and $(M, \mathbf{a}) \equiv_F (M, \mathbf{b})$ then there is an automorphism of M mapping \mathbf{a} to \mathbf{b} . If we pick α such that every ψ in F has quantifier rank below α and $(M, \mathbf{a}) \sim_\alpha (M, \mathbf{b})$, then $(M, \mathbf{a}) \sim_\beta (M, \mathbf{b})$ for all β . Thus the Scott rank of ϕ is at most α .

If F is the smallest fragment containing ϕ , every formula in F has Scott rank below $\text{qr}(\phi) + \omega$, so this is an upper bound on the Scott rank. $\square_{8.13}$

Proof of Theorem 8.9. First note that if α is a bound on the Scott rank of models of ϕ , then any two countable models M and N of ϕ are isomorphic if and only if we can do a back-and forth construction using \sim_α . Thus by Nadel's Theorems, a sentence ϕ is complete if and only if

- i) $(\exists M)M \leq_{\text{hyp}} \phi \wedge M \models \phi$ and
- ii) $\exists \alpha$ recursive in ϕ such that for all $M, N \models \phi$ if $\mathbf{a} \in M, \mathbf{b} \in N$ and $(M, \mathbf{a}) \sim_\alpha (N, \mathbf{b})$, then for all $c \in M$ there is $d \in N$ such that $(M, \mathbf{a}, c) \sim_\alpha (N, \mathbf{b}, d)$.

i) is easily seen to be Π_1^1 , using Fact 5.4.

ii) is equivalent to $\forall M, N \models \phi (\exists R, \exists z)z \leq_{\text{hyp}} \phi, R \in WO^*$ and z is an R -analysis of M and N and there is an n such that if $\mathbf{a}, c \in M, \mathbf{b} \in N$ with $(n, \mathbf{a}, \mathbf{b}) \in z$, then there is $d \in N$ such that $(n, \mathbf{a}, c, \mathbf{b}, d) \in z$. This is also Π_1^1 , again using Fact 5.4.

Finally, to each linear order \prec of ω we will assign an $L_{\omega_1, \omega}$ sentence ϕ_\prec such that \prec is a well order if and only if ϕ_\prec is complete. This will show that $\{\phi : \phi \text{ is complete}\}$ is Π_1^1 -complete.

The vocabulary τ is $\{P_n : n \in \omega\}$ where P_n is a unary predicate.

- We say that every element is in some P_n .
- We say that each P_n is infinite and that if $n \prec m$, then $P_n \subset P_m$ and $P_m \setminus P_n$ is infinite.
- Moreover if $\forall m \prec n \exists k m \prec k \prec n$, then we also say that $P_n \setminus \bigcup_{m \prec n} P_m$ is infinite.

If \prec is a well ordering, then ϕ_\prec is \aleph_0 -categorical as for each n we just put \aleph_0 elements in each $P_n \setminus \bigcup_{m \prec n} P_m$.

On the other hand if $n_0 \succ n_1 \succ \dots$ is an infinite descending chain let $X = \{m : m \prec n_i \text{ for all } i\}$. We can put any number of elements in

$$\bigcap_{i=0}^{\infty} P_{n_i} \setminus \bigcup_{m \in X} P_m,$$

so ϕ_\prec is not complete. □_{8.9}

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