Math 300 Writing for Mathematics Spring 2003

Mathematics for Essay 2

Consider the infinite series:
\[ 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots \]  
(1)

This is the case, \( p = 2 \) of the \( p \)-series:
\[ \sum_{n=1}^{\infty} \frac{1}{n^p}. \]

We know from our discussion in Essay 1 that
\[ \sum_{n=1}^{\infty} \frac{1}{n^2} < 2. \]

We noticed in Essay 1 that these are the areas of a collection of squares that fit in a 1 by 2 rectangle with a lot left over. We would like to get a better approximation to the actual value.

Sketch the graph of the function \( f(x) = \frac{1}{x^2} \). Note that there are disjoint rectangles with area \( \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25} \) contained in the region bounded by the x-axis, \( f(x) = \frac{1}{x^2} \), and the line \( x = 1 \). (Take the rectangle formed by going one unit to the left from the point \( (n, \frac{1}{n^2}) \) to get area \( \frac{1}{n^2} \).) So
\[ \int_{1}^{\infty} \frac{1}{x^2} \, dx > \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots. \]

More generally
\[ \int_{m}^{\infty} \frac{1}{x^2} \, dx > \frac{1}{(m+1)^2} + \frac{1}{(m+2)^2} + \frac{1}{(m+3)^2} + \cdots. \]

So for each \( m \), we can get an upper bound on
\[ 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots + \int_{m}^{\infty} \frac{1}{x^2} \, dx. \]

The first of these term is finite; find it with a calculator. On the other hand
\[ \int_{m}^{\infty} \frac{1}{x^2} \, dx = -\frac{1}{x} \bigg|_{m}^{\infty} = \lim_{b \to \infty} \frac{-1}{x} \bigg|_{m}^{b} = \lim_{b \to \infty} \left( \frac{-1}{b} - \frac{-1}{m} \right) = \frac{1}{m}. \]

For example, taking \( m = 2 \), my estimate is \((1 + 1/4) + 1/2 = 13/4\), for \( m = 3 \) it is \((1 + 1/4 + 1/9) + 1/3 \) which is approximately 1.69.

Now to estimate the volumes repeat this process but with \( g(x) = \frac{1}{x^3} \) replacing \( f(x) \). Remarkably, the exact value for \( \sum_{n=1}^{\infty} \frac{1}{n^3} \) is \( \frac{\pi^3}{6} \) while the exact value of \( \sum_{n=1}^{\infty} \frac{1}{n^4} \) is unknown.

See www.mat.bham.ac.uk/C.J.Sangwin/Teaching/pus/infsersup.pdf