The complex numbers and complex exponentiation Why Infinitary Logic is necessary!

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SELF CONSCIOUS MATHEMATICS

- A signature L is a collection of relation and function symbols.

- A structure for that signature (L-structure) is a set with an interpretation for each of those symbols.

- The first order language ($L_{\omega,\omega}$) associated with L is the least set of formulas containing the atomic L-formulas and closed under **finite** Boolean operations and quantification over finitely many individuals.

- The $L_{\omega_1,\omega}$ language associated with L is the least set of formulas containing the atomic L-formulas and closed under **countable** Boolean operations and quantification over finitely many individuals.

Model Theory and Number Theory

Prior to 1980: Use of basic model theoretic notions: compactness, quantifier elimination: Ax-Kochen-Ershov

Later: Increasing use of sophisticated first order model theory: stability theory; Shelah's orthogonality calculus; o-minimality: Hrushovski's proof of geometric Mordell-Lang.

ALGEBRAICALLY CLOSED FIELDS

Fundamental structure of Algebraic Geometry

Axioms for fields of fixed characteristic and

$$(\forall a_1, \ldots a_n)(\exists y)\Sigma a_i y^i = 0$$

MODEL THEORETIC FUNDAMENTAL

The theory T_p of algebraically closed fields of fixed characteristic has exactly one model in each uncountable cardinality. (Steinitz)

That is, T_p is *categorical* in each uncountable cardinality

Categorical Structures

I. $(\mathcal{C}, =)$

IIa. $(\mathcal{C}, +, =)$ vector spaces over Q.

IIb. $(\mathcal{C}, \times, =)$

MORLEY'S THEOREM

Theorem 1 *If a countable first order theory is categorical in one uncountable cardinal it is categorical in all uncountable cardinals.*

Morley Rank

‘. . . what makes his paper seminal are its new techniques, which involve a systematic study of Stone spaces of Boolean algebras of definable sets, called type spaces. For the theories under consideration, these type spaces admit a Cantor Bendixson analysis, yielding the key notions of Morley rank and ω-stability.’

Citation awarding Michael Morley the 2003 Steele prize for seminal paper.

DECIDABILITY

Corollary 2 *The set of sentences true in algebraically closed fields of a fixed characteristic is decidable.*

LINDSTROMS'S LITTLE THEOREM

Theorem 3 *If* T *is* $\forall\exists$*-axiomatizable and categorical in some infinite cardinality then* T *is model complete.*

COROLLARIES

Corollary 4 *The theory of algebraically closed fields admits elimination of quantifiers.*

Corollary 5 (Tarski, Chevalley) *The projection of a constructible set is constructible.*

STRONGLY MINIMAL I

Definition 6 M *is* strongly minimal *if every first order definable subset of any elementary extension* M' *of* M *is finite or cofinite.*

Every strongly minimal set is categorical in all uncountable powers.

The complex field is strongly minimal.

GÖDEL PHENOMENA

It follows from Gödels work in the 30's that:

1. The collection of sentences true in $(Z, +, \cdot, 0, 1)$ is undecidable.

2. There are definable subsets of $(Z, +, \cdot, 0, 1)$ which require arbitrarily many alternations of quantifiers. (Wild)

COMPLEX EXPONENTIATION

Consider the structure $(C, +, \cdot, e^x, 0, 1)$.

It is Godelian.

The integers are defined as $\{a : e^a = 1\}$.

The first order theory is undecidable and ‘wild’.

ZILBER'S INSIGHT

Maybe Z is the source of all the difficulty. Fix Z by adding the axiom:

$$(\forall x)e^x = 1 \to \bigvee_{n \in Z} x = 2n\pi.$$

REPRISE

The first order theory of the complex field is categorical and admits quantifier elimination.

Model theoretic approaches based on Shelah's theory of orthogonality have led to advances such as Hrushovski's proof of the geometric Mordell-Lang conjecture.

The first order theory of complex exponentiation is model theoretically intractable;

We now explore infinitary approaches.

GEOMETRIES

Definition. A pregeometry is a set G together with a dependence relation

$$cl : \mathcal{P}(G) \to \mathcal{P}(G)$$

satisfying the following axioms.

A1. $cl(X) = \bigcup\{cl(X') : \; X' \subseteq_{fin} X\}$

A2. $X \subseteq cl(X)$

A3. $cl(cl(X)) = cl(X)$

A4. If $a \in cl(Xb)$ and $a \notin cl(X)$, then $b \in cl(Xa)$.

If points are closed the structure is called a geometry.

CLASSSIFYING GEOMETRIES

Geometries are classified as: trivial, locally modular, non-locally modular.

Zilber had conjectured that each non-locally modular geometry of a strongly minimal set was ‘essentially’ the geometry of an algebraically closed field.

We will study Hrushovski’s construction which gave counterexamples to this conjecture and maybe much more.

STRONGLY MINIMAL II

$a \in$ acl(B) if $\phi(a, \bar{b})$ and $\phi(x, \bar{b})$ has only finitely many solutions.

A complete theory T is strongly minimal if and only if it has infinite models and

1. algebraic closure induces a pregeometry on models of T;

2. any bijection between acl-bases for models of T extends to an isomorphism of the models

QUASIMINIMALITY I

Trial Definition M is *'quasiminimal'* if every first order ($L_{\omega_1,\omega}$?) definable subset of M is countable or cocountable.

$a \in$ acl′(X) if there is a first order formula with **countably many** solutions over X which is satisfied by a.

Exercise ? If f takes X to Y is an elementary isomorphism, f extends to an elementary isomorphism from acl′(X) to acl′(Y).

QUASIMINIMALITY II

QUASIMINIMAL EXCELLENCE

A class $(\boldsymbol{K}, \mathrm{cl})$ is *quasiminimal excellent* if it admits a combinatorial geometry which satisfies on each $M \in \boldsymbol{K}$

there is a unique type of a basis,

a technical homogeneity condition:
$\aleph_0$-homogeneity over $\emptyset$ and over models.

and the 'excellence condition' which follows.

In the following definition it is essential that $\subset$ be understood as *proper* subset.

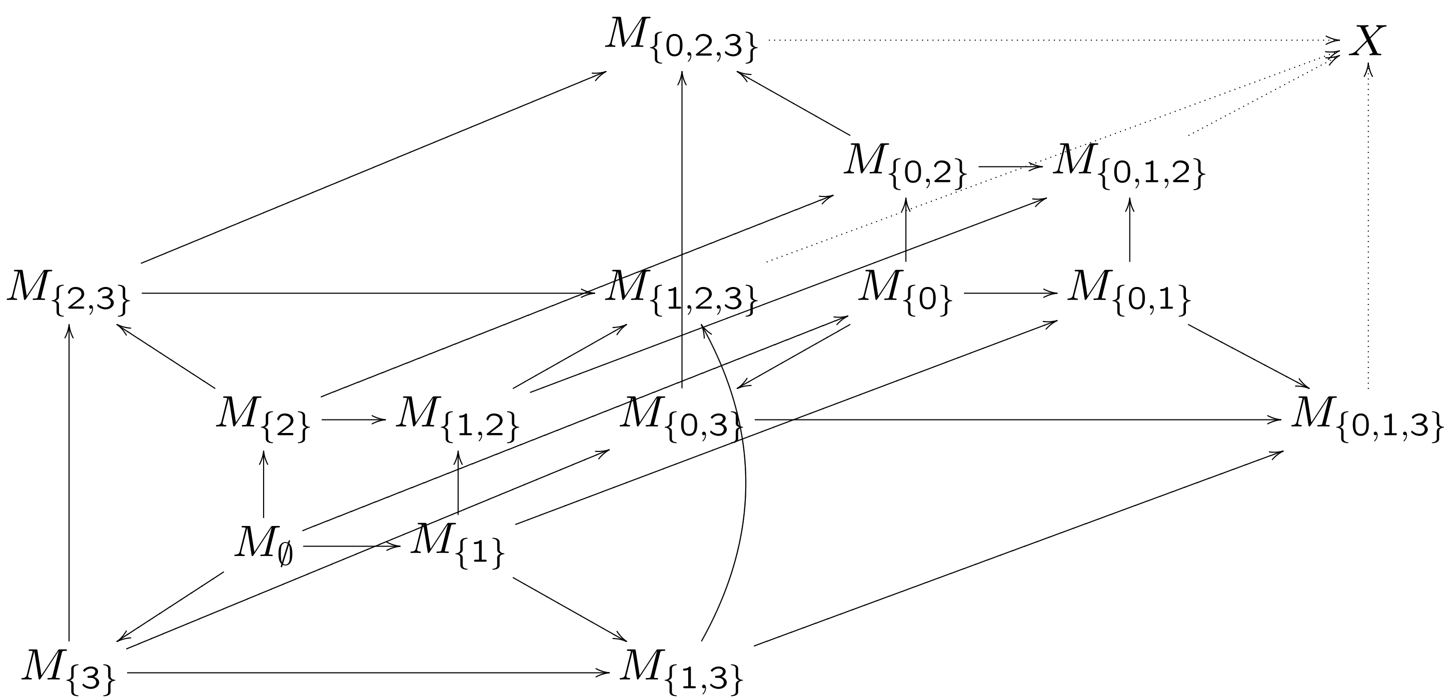
Definition 7 *1. For any* Y, $\mathrm{cl}^{-}(Y) = \bigcup_{X \subset Y} \mathrm{cl}(X)$.

2. *We call* C *(the union of)* an n-dimensional cl-independent system *if* $C = \mathrm{cl}^{-}(Z)$ *and* Z *is an independent set of cardinality* n.

n-AMALGAMATION

$$
\begin{tikzcd}
 & & & M_{\{1,3\}} \arrow[rr, dotted] & & X \\
M_{\{1\}} \arrow[rr] \arrow[urrr] & & M_{\{1,2\}} \arrow[urrr, dotted] & & & \\
 & & & M_{\{3\}} \arrow[uu] \arrow[rr] & & M_{\{2,3\}} \arrow[uu, dotted] \\
M_{\emptyset} \arrow[uu] \arrow[rr] \arrow[urrr] & & M_{\{2\}} \arrow[uu] \arrow[urrr] & & &
\end{tikzcd}
$$

4-excellence



$\mathcal{P}(n)$ powerset of n, let $\mathsf{p}^{-}(n) := \mathcal{P}(n) - \{n\}$.

Let say $\mathrm{tp}_{\mathrm{qf}}(X/C)$ is *defined over* the finite C_0 contained in C if it is determined by its restriction to C_0.

[Quasiminimal Excellence] Let $G \subseteq H \in \boldsymbol{K}$ with G empty or in $\boldsymbol{K}$. Suppose $Z \subset H - G$ is an n-dimensional independent system, $C = \mathrm{cl}^{-}(Z)$, and X is a finite subset of $\mathrm{cl}(Z)$. Then there is a finite C_0 contained in C such that $\mathrm{tp}_{\mathrm{qf}}(X/C)$ is defined over C_0.

EXCELLENCE IMPLIES CATEGORICITY

Excellence implies by a direct limit argument:

Lemma 8 *An isomorphism between independent* X *and* Y *extends to an isomorphism of* $\mathrm{cl}(X)$ *and* $\mathrm{cl}(Y)$.

This gives categoricity in all uncountable powers if the closure of each finite set is countable.

CATEGORICITY

Theorem Suppose the quasiminimal excellent (I-IV) class $\boldsymbol{K}$ is axiomatized by a sentence Σ of $L_{\omega_1,\omega}$, and the relations $y \in \mathrm{cl}(x_1, \ldots x_n)$ are $L_{\omega_1,\omega}$-definable.

Then, for any infinite κ there is a unique structure in $\boldsymbol{K}$ of cardinality κ which satisfies the countable closure property.

NOTE BENE: The categorical class could be axiomatized in $L_{\omega_1,\omega}(Q)$. But, the categoricity result does not depend on any such axiomatization.

DIMENSION FUNCTIONS

Let $\boldsymbol{K}_0$ be a class of substuctures closed under submodel.

A predimension is a function δ mapping finite subsets of members of $\boldsymbol{K}$ into the integers such that:

$$\delta(XY) \leq \delta(X) + \delta(Y) - \delta(X \cap Y).$$

For each $N \in \boldsymbol{K}$ and finite $X \subseteq N$, the *dimension* of X in N is

$$d_N(X) = \min\{\delta(X') : X \subseteq X' \subseteq_\omega N\}.$$

The dimension function

$$d : \{X : \ X \subseteq_{fin} G\} \to \mathbf{N}$$

satisfies the axioms:

D1. $d(XY) + d(X \cap Y) \leq d(X) + d(Y)$

D2. $X \subseteq Y \ \Rightarrow \ d(X) \leq d(Y)$.

THE GEOMETRY

Definition 9

For A, b *contained* M, $b \in \mathrm{cl}(A)$ *if* $d_M(bA) = d_M(A)$.

Naturally we can extend to closures of infinite sets by imposing finite character. If d satsfies:

D3 $d(X) \leq |X|$.

we get a full combinatorial (pre)-geometry with exchange.

ZILBER'S PROGRAM FOR $(\mathcal{C}, +, \cdot, \exp)$

Goal: Realize $(\mathcal{C}, +, \cdot, \exp)$ as a model of an $L_{\omega_1,\omega}(Q)$-sentence discovered by the Hrushovski construction.

A. Expand $(\mathcal{C}, +, \cdot)$ by a unary function which behaves like exponentiation using a Hrushovski-like dimension function. Prove some $L_{\omega_1,\omega}(Q)$-sentence Σ is categorical and has quantifier elimination.

B. Prove $(\mathcal{C}, +, \cdot, \exp)$ is a model of the sentence Σ found in Objective A.

THE AXIOMS

$L = \{+, \cdot, E, 0, 1\}$

$(K, +, \cdot, E) \models \Sigma$ if

K is an algebraically closed field of characteristic 0.

E is a homomorphism from $(K, +)$ onto $(K^x, \cdot)$ and there is $\nu \in K$ transcendental over $\mathbb{Q}$ with $\ker E = \nu Z$.

E is a pseudo-exponential

K is strongly exponentially algebraically closed.

PSEUDO-EXPONENTIATION

E is a **pseudo-exponential** if for any n linearly independent elements over $\mathbb{Q}$, $\{z_1, \ldots z_n\}$

$$\mathrm{td}(z_1, \ldots z_n, E(z_1), \ldots E(z_n)) \geq n.$$

Schanuel conjectured that true exponentiation satisfies this equation.

ABSTRACT SCHANUEL

For a finite subset X of an algebraically closed field k with a partial exponential function. Let

$$\delta(X) = \mathrm{td}(X \cup E(X)) - \mathrm{ld}(X).$$

Apply the Hrushovski construction to the collection of (k, E) with $\delta(X) \geq 0$ for all finite $X \subset k$. That is, those which satisfy the abstract Schanuel condition.

The result is a quasiminimal excellent class.

ALGEBRA FOR OBJECTIVE A

Conjecture on intersection of tori

Given a variety $W \subseteq \mathcal{C}^{n+k}$ defined over $\mathbb{Q}$, and a coset $T \subseteq (\mathcal{C}^*)^n$ of a torus.

An infinite irreducible component S of $W(\bar{b}) \cap T$ is **atypical** if

$$d_f S - \dim T > d_f W(\bar{b}) - n.$$

Theorem 10 *There is a finite set A of nonzero elements of $\mathcal{Z}^n$,so that if S is an atypical component of $W \cap T$ then for some $\overline{m} \in A$ and some γ from $\mathcal{C}$, every element of S satisfies $\overline{x}^{\overline{m}} = \gamma$.*

Using the true CIT, the abstract Schanuel condition becomes a first order property.

Replacing $\mathcal{C}$ by a semialgebraic variety gives the conjectured full CIT, which implies Manin-Mumford and more.

CHOOSING ROOTS

Definition 11 *A* multiplicatively closed divisible subgroup *associated with* $a \in \mathcal{C}^*$, *is a* **choice** *of a multiplicative subgroup isomorphic to* $\mathbb{Q}$ *containing* a *.*

Definition 12 $b_1^{\frac{1}{m}} \in b_1^{\mathbb{Q}}, \ldots b_\ell^{\frac{1}{m}} \in b_\ell^{\mathbb{Q}} \subset \mathcal{C}^*$, determine the isomorphism type *of* $b_1^{\mathbb{Q}}, \ldots b_\ell^{\mathbb{Q}} \subset \mathcal{C}^*$ *over* F *if given subgroups of the form* $c_1^{\mathbb{Q}}, \ldots c_\ell^{\mathbb{Q}} \subset \mathcal{C}^*$ *and* ϕ_m *such that*

$$\phi_m : F(b_1^{\frac{1}{m}} \ldots b_\ell^{\frac{1}{m}}) \to F(c_1^{\frac{1}{m}} \ldots c_\ell^{\frac{1}{m}})$$

is a field isomorphism it extends to

$$\phi_\infty : F(b_1^{\mathbb{Q}}, \ldots b_\ell^{\mathbb{Q}}) \to F(c_1^{\mathbb{Q}}, \ldots c_\ell^{\mathbb{Q}}).$$

Theorem 13 (thumbtack lemma)

For any $b_1, \ldots b_\ell \subset \mathcal{C}^*$, *there exists an* m *such that* $b_1^{\frac{1}{m}} \in b_1^{\mathbb{Q}}, \ldots b_\ell^{\frac{1}{m}} \in b_\ell^{\mathbb{Q}} \subset \mathcal{C}^*$, determine the isomorphism type *of* $b_1^{\mathbb{Q}}, \ldots b_\ell^{\mathbb{Q}} \subset \mathcal{C}^*$ *over* F.

The Thumbtack Lemma implies that $\boldsymbol{K}$ satisfies the homogeneity conditions and ‘excellence’.

F can be the acf of Q or a number field, or an independent system of algebraically closed fields. If $\mathcal{C}$ is replaced by a semi-abelian variety, these differences matter.

TOWARDS EXISTENTIAL CLOSURE

Given $V \subseteq K^{2n}$ we might want to find $z_1, \ldots, z_n$ with

$(z_1, \ldots z_n, E(z_1), \ldots E(z_n)) \in V$.

Schanuel's conjecture prevents this for 'small' varieties. We want to say this is the only obstruction.

NORMAL VARIETY

Let $G^n(F) = F^n \times (F^*)^n$.

If M is a $k \times n$ integer matrix,

$[M] : G^n(F) \to G^n(F)$ is the homomorphism taking $\langle \overline{a}, \overline{b} \rangle$ to $\langle M\overline{a}, \overline{b}^M \rangle$. Act additively on first n coordinates, multiplicatively on the last n. V^M is image of V under M.

V is *normal* if for any rank k matrix M, $\dim V^M \geq k$.

FREE VARIETIES

Let $V(\overline{x}, \overline{y})$ be a variety in $2n$ variables. $\mathrm{pr}_{\overline{x}}\mathrm{V}$ is the projection on $\overline{x}$, $\mathrm{pr}_{\overline{y}}\mathrm{V}$ is the projection on $\overline{y}$

V contained in F^{2n}, exp-definable over C is *absolutely free of additive dependencies* if for a generic realization $\overline{a} \in \mathrm{pr}_{\overline{x}}\mathrm{V}$, $\overline{a}$ is additively linearly independent over acl(C).

V contained in F^{2n}, exp-definable over C is *absolutely free of multiplicative dependencies* if for a generic realization $\overline{b} \in \mathrm{pr}_{\overline{y}}\mathrm{V}$, $\overline{b}$ is multiplicatively linearly independent over acl(C).

STRONG EXPONENTIAL CLOSURE

Let $V \subseteq G_n(K)$ be free, normal and irreducible.

For every finite A, there is $(\overline{z}, E(\overline{z})) \in V$ which is generic for A.

This is $L_{\omega_1,\omega}$-expressible; using uniform CIT (Holland, Zilber) it is first order.

COUNTABLE CLOSURE

Under the geometry imposed by

$$\delta(X) = \mathrm{td}(\overline{x}, \mathrm{E(xbar)} - \mathrm{ld(X)}$$

, the Schanuel condition.

the closure of a finite set is countable.

OBJECTIVE A

Corollary. The models of Σ with countable closure are categorical in all uncountable powers. This class is $L_{\omega_1,\omega}(Q)$-axiomatizable.

Objective B
GENUINE EXPONENTIATION?

Schanuel's conjecture: If $x_1, \ldots x_n$ are $\mathbb{Q}$-linearly independent complex numbers then $x_1, \ldots x_n, e^{x_1}, \ldots e^{x_n}$ has transcendence degree at least n over $\mathbb{Q}$.

Zilber showed:

Theorem. If Schanuel holds in $\mathcal{C}$ and if the (strong) existential closure axioms hold in $\mathcal{C}$, then $(\mathcal{C}, +, \cdot, \exp) \in \mathrm{E}C^*_{st}$. $(\mathcal{C}, +, \cdot, \exp)$ has the countable closure property.

VERIFYING EXPONENTIAL COMPLETENESS

We want:

For any free normal V given by $p(z_1, \dots z_n, w_1, \dots w_n) = 0$, with $p \in \mathbb{Q}[z_1, \dots z_n, w_1, \dots w_n]$, and any finite A there is a solution satisfying

$$(z_1, \dots z_n, E(z_1), \dots E(z_n)) \in V.$$

and $z_1, \dots z_n, E(z_1), \dots E(z_n)$ is generic for A.

VERIFIED EXPONENTIAL COMPLETENESS

Marker has proved.

Assume Schanuel.

If $p(x, y) \in \mathbb{Q}[x, y]$ and depends on both x and y then it has infinitely many algebraically independent solutions.

This verifies the n-variable conjecture for $n = 1$ with strong restrictions on the coefficients.

The proof is a three or four page argument using Hadamard factorization.

MODEL THEORETIC CONTEXT

Any κ-categorical sentence of $L_{\omega_1,\omega}$ can be replaced (for categoricity purposes) by considering the atomic models of a first order theory. ($EC(T, Atomic)$-class)

Shelah defined a notion of excellence; Zilber's is the 'rank one' case.

Theorem 14 (Shelah 1983) *If $\boldsymbol{K}$ is an excellent $EC(T, Atomic)$-class then if it categorical in one uncountable cardinal, it is categorical in all uncountable cardinals.*

Theorem 15 (Shelah 1983) *If $2^{\aleph_n} < 2^{\aleph_{n+1}}$ and an $EC(T, Atomic)$-class $\boldsymbol{K}$ is categorical in all $\aleph_n$ for all $n < \omega$, then it is excellent.*

An example with Hart shows the infinitely many instances of categoricity are necessary.

The categoricity arguments were ‘Morley-style’. Lessmann has given ‘Baldwin-Lachlan’ style proofs - showing models prime over quasiminimal sets.

First Order to infinitary

Strongly minimal is to first order

as

Quasiminimal excellent is to $L_{\omega_1,\omega}$.

But the analogy slips with consideration of $L_{\omega_1,\omega}(Q)$.

UNIVERSAL COVERS

When is the exact sequence:

$$0 \to Z \to V \to A \to 0. \tag{1}$$

categorical where V is a $\mathbb{Q}$ vector space and A is a semi-abelian variety.

Zilber showed 'the thumbtack lemma' is sufficient.

(and true – when $A = (\mathcal{C}, \cdot)$.

CONVERSELY

Applying Shelah's theorem, Zilber showed: if

$$0 \to Z \to V \to A \to 0. \tag{2}$$

is categorical up to $\aleph_\omega$ then the arithmetic statements of the 'thumbtack lemma' are true for A.

MESSAGE

The analysis of number theoretic problems using infinitary logic provides exciting opportunities for continuing the almost 100 year interaction between model theory and number theory.

GO FORTH AND MULTIPLY

http://www2.math.uic.edu/ jbaldwin/model.html