Geometry, the Common Core, and Proof

John T. Baldwin, Andreas Mueller

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Outline

1. The motivating problem
2. Euclidean Axioms and Diagrams
3. The Rusty compass
4. Congruence
5. Definitions
Agenda

1. G-C0-1 – Context.
2. Activity: Divide a line into n pieces - with string; via construction
3. Reflection activity (geometry/ proof/ definition/ common core)
4. mini-lecture Axioms and Definitions in Euclid
6. Diagrams and proofs
7. lunch/Discussion: How do these differ from axioms in high school texts
8. Activity - rusty compass theorem (30 min)
9. congruence as a basic notion; SSS
10. Discussion of G-C0 1
G-C01

Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

Why is word undefined in this standard? What and how do we know about the undefined notions?
Activity: Dividing a line into n-parts: Informal

See Handout
Activity: Dividing a line into n-parts: Construction

Here is a procedure to divide a line into \( n \) equal segments.

1. Given a line segment \( AB \).
2. Draw a line through \( A \) different from \( AB \) and lay off sequentially \( n \) equal segments on that line, with end points \( A_1, A_2, \ldots \). Call the last point \( D \).
3. Construct \( C \) on the opposite side of \( AB \) from \( D \) so that \( AC \cong BD \) and \( CB \cong AD \).
4. Lay off sequentially \( n \) equal segments on that line, with end points \( C, B_1, B_2, \ldots B \).
5. Draw lines \( A_iB_i \).
6. The point \( C_i \) where \( C_i \) is the intersection of \( A_iB_i \) with \( AB \) are the required points dividing \( AB \) into \( n \) equal segments.
Activity: Dividing a line into n-parts: Diagram

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The motivating problem

Euclidean Axioms and Diagrams

The Rusty compass

Congruence

Definitions
Why did it work?

Do exercise 2.2 in text.
Reflection: Constructions, proofs, definitions

See activity worksheet.
These common notions or *axioms* of Euclid apply equally well to geometry or numbers. We won’t make a big deal over the difference between ‘axiom’ and ‘postulate’.

Common notion 1. Things which equal the same thing also equal one another.
Common notion 2. If equals are added to equals, then the wholes are equal.
Common notion 3. If equals are subtracted from equals, then the remainders are equal.
Common notion 4. Things which coincide with one another equal one another.
Common notion 5. The whole is greater than the part.
Modern (new math) text books make a big deal about the difference between congruence and equality. Numbers are central - so equalities are only between numbers while line segments or figures are congruent.

**Geometry before Number**

Euclid did not have numbers as distinct objects - He’d say line segments are congruent where we’d say have the same length.

So equality can be replaced by congruence in understanding the common notions.
Euclid’s first 3 axioms in modern language

1. **Axiom 1** Given any two points there is a line segment connecting them.

2. **Axiom 2** Any line segment can be extended indefinitely (in either direction).

3. **Axiom 3** Given a point and any segment there is a circle with that point as center whose radius is the same length as the segment.
What are these axioms telling us?

Discussion:
What are these axioms telling us?

Discussion:
implicit definition
Activity: Construct an equilateral triangle

Do it with straightedge and compass.
Activity: Construct an equilateral triangle

Do it with straightedge and compass.
Does anyone have any doubts?
Activity: Construct an equilateral triangle

Do it with straightedge and compass.

Does anyone have any doubts?

Suppose the base line is between \((-1, 0)\) and \((1, 0)\). What are coordinates of the vertex?
Activity: Construct an equilateral triangle

Do it with straightedge and compass.

Does anyone have any doubts?

Suppose the base line is between $(-1,0)$ and $(1,0)$. What are coordinates of the vertex?

Suppose I just take a plane with rational coordinates. What’s up?
Axioms on the intersection of circles

1. **Axiom 3’** If a circle is drawn with radius $AB$ and center $A$, then it intersects any line through $A$ other than $AB$ in two points $C$ and $D$, one on each side of the line $AB$.

2. **Axiom 3’’** If from points $A$ and $B$, two circles $C_1$ and $C_2$ are drawn so that each circle contains points both in the interior and in the exterior of the other, then they intersect in two points, each on opposites sides of $AB$.” then they intersect in two points, one on each side of $AB$.

We used ‘inexact’ properties to phrase the axiom. It could be done by more detailed conditions on the radii.
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Hilbert et al banished the diagram from formal mathematics. This requires complicated axioms about betweenness which have never successfully been taught to high school students.
The Cure is worse than the disease

The ‘new math’ tried to replace the diagram with algebra. http://www.glencoe.com/sec/math/studytools/cgi-bin/msgQuiz.php4?isbn=0-07-829637-4&chapter=2&lesson=7&quizType=1&headerFile=6&state=il
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Why does it take six steps to show: If two line segments have the same length and equal line segments are taken away from each, the resulting segments have the same length.
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Why does it take six steps to show: If two line segments have the same length and equal line segments are taken away from each, the resulting segments have the same length.

Common Notions
Inexact properties

Properties that are **not** changed by minor variations in the diagram.

subsegment, inclusion of one figure in another, two lines intersect, betweenness

Exact properties

Properties that **can be** changed by minor variations in the diagram.

Whether a curve is a straight line, congruence, a point is on a line

We can rely on inexact properties from the diagram. We must write exact properties in the text.
The fly in the ointment

Depending on the particular diagram that you draw, after a construction, the diagram may have different inexact properties. The solution is case analysis. We will spend time on this later in the workshop.
Choice of Axioms matters

Students learned to prove from Euclid.
Choice of Axioms matters

Students learned to prove from Euclid.
Student are buried by trivialities and don’t learn to prove from Moise-Birkhoff.
We will try in this workshop to develop ways of using diagrams in a semi-formal way to support the line-by-line argument. This worked for 2000 years. But it hasn’t been rigorously formulated nor implemented with modern students. (Miller book)
Activity: Euclid Proposition 2

Understanding the reason for and proof of Proposition 2.
What is congruence

Undefined concept: Congruence.

of angles, of line segments
We have axioms about congruence.

The common notions apply. In the Common Notions ‘congruence’ can be substituted for equal.
Is CPTPC a definition, a proof, a theorem, a postulate, an undefined notion?
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**Definition: Figure congruence**

**CCSS G-C0-7** Two figures are congruent if there is a way to make the sides and angles correspond so that:
Each pair of corresponding angles are congruent.
Each pair of corresponding sides are congruent.
Is CPTPC a definition, a proof, a theorem, a postulate, an undefined notion?

**Definition: Figure congruence**

**CCSS G-C0-7** Two figures are congruent if there is a way to make the sides and angles correspond so that:
- Each pair of corresponding angles are congruent.
- Each pair of corresponding sides are congruent.

Note this is really about polygons. And we haven’t defined angle yet.
The triangle congruence postulate: SSS

CCSS G-C0-8 Let $ABC$ and $A'B'C'$ be triangles with $AB \cong A'B'$ and $AC \cong A'C'$ and $BC \cong B'C'$ then $\triangle ABC \cong \triangle A'B'C'$
Homework Challenge

PROVE:

**Theorem SAS**

**CCSS G-C0-8, G-C0-10** Let \(ABC\) and \(A'B'C'\) be triangles with \(AB \cong A'B'\) and \(AC \cong A'C'\) and \(\angle CAB \cong \angle C'A'B'\) then \(\triangle ABC \cong \triangle A'B'C'\)

We will discuss the proof next week.
The motivating problem

Euclidean Axioms and Diagrams

The Rusty compass

Congruence Definitions

G-C01

Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

We have seen how axioms are necessary to specify undefined notions. Now let’s address the standard.
Defining the important concepts

Using the undefined notions

of point, line, congruent line segment, congruent angles,

Define

line segment

1. group 1 angle, right angle, perpendicular line,
2. group 2 circle, arc
3. parallel line

Explain each notion with physical models or transformations. A group can use the notions defined by groups with smaller numbers.
Defining the measure of an angle

For this we need the undefined notion of ‘length of an arc’ (i.e. congruence of circles)