Geometry, the Common Core, and Proof

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November 28, 2012

Outline

Agenda

1. G-SRT4 – Context. Proving theorems about similarity
2. Parallelograms - napoleon's theorem
3. Area: informally and formallly
4. Areas of parallelograms and triangles
5. a bit more on parallel lines
6. mini-lecture Geometry vrs arithmetic
7. Introducing Arithmetic
   i) functions
   ii) defining addition: segments, points
   iii) defining multiplication: segments, points
8. lunch/Discussion: How are numbers and geometry being treated than in high school texts
9. Interlude on circles
   i) How many points determine a circle
   ii) Diagrams and proofs
10. Proving that there is a field

Logistics

How many people have had a chance to spend any time with the materials on line?
Is there a way they could be more helpful?

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Does anyone want a vegetarian sandwich for lunch?

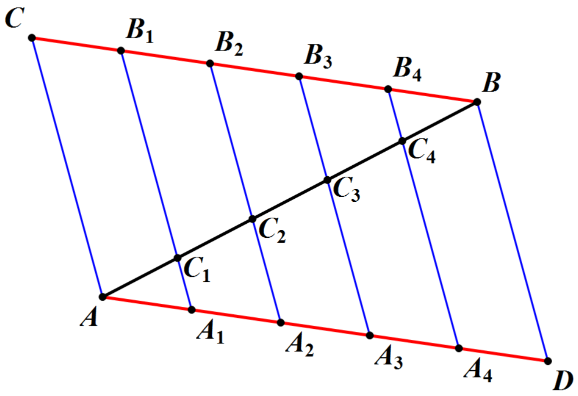
Common Core

G-SRT: Prove theorems involving similarity

4. Prove theorems about triangles. Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.

This is a particularly hard standard to understand. What are we supposed to assume when we give these proofs.
Should the result be proved when the ratio of the side length is irrational?

Activity: Dividing a line into n-parts: Diagram



Sketch of Proof

1. $CBDA$ is a parallelogram. (today soon)
2. Each B_nBDA_n is a parallelogram. (next week)
3. Therefore all segments C_nC_{n+1} have the same length.

Step 3 is the hard part. We will spend most of today on it. The key is:

Side-splitter Theorem

Theorem

Euclid VI.2 **CCSS G-SRT.4** If a line is drawn parallel to the base of triangle the corresponding sides of the two resulting triangles are proportional and conversely.

CME geometry calls this the 'side-splitter theorem' on pages 313 and 315 of CME geometry.
Two steps:

1. Understand the area of triangle geometrically.
2. Transfer this to the formula $A = \frac{bh}{2}$.

The proof uses elementary algebra and numbers; it is formulated in a very different way from Euclid. To prove the side-splitter theorem next week, we have to introduce numbers.

Activity: Midpoints of sides of Quadrilaterals

Let ABCD be an arbitrary quadrilateral? Let DEFG be the midpoints of the sides. What can you say about the quadrilateral DEFG?

I called this Napoleon's Theorem in class. That was a mistake; there is somewhat similar proposition called Napoleon's theorem.

Activity: Conditions for a parallelogram

Theorem

If the opposite sides of a quadrilateral are congruent, the quadrilateral is a parallelogram.

Prove this theorem. What is the key step that is also needed for Napoleon's theorem?

Activity: Triangles with the same area: Informal

See Handout

Activity: Scissor Congruence

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174-175 from CME

scissors congruence

What does scissor's congruence mean?

scissors congruence

What does scissor's congruence mean?

If it is possible to cut one figure into a finite number of pieces and rearrange them to get the other, then they are scissor congruent.

Hilbert defines:

Two polygons are equidecomposable if they can be decomposed into a finite number of triangles that are congruent in pairs.

Two simple polygons P and Q are equicomplementable if is possible to add to them a finite number of pairs P_i, Q_i of equidecomposable polygons so that the disjoint union of Q with the Q_i is congruent to disjoint union of P with the P_i

Activity: Prove Euclid I.35

Theorem

Parallelograms on the same base and in the same parallels have the same area.

Restate in modern English; prove the theorem.
in groups (10 minutes)

Why did it work?

What are the properties of Area that are needed for this argument?
Did you use equidecomposable or equicomplementable polygons?

Axioms for Area

1. Congruent figures have the same area.
2. The area of two 'disjoint' polygons (i.e. meet only in a point or along an edge) is the sum of the two areas of the polygons.
3. Two figures that scissor-congruent have the same area.
4. The area of a rectangle whose base has length b and altitude is h is bh.

Deepening understanding across grade levels

6th grade standard (CCSS 6.G.1)

Solve real-world and mathematical problems involving area, surface area, and volume. 1. Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

Do your students understand this?

Common Notions

These common notions or *axioms* of Euclid apply equally well to geometry or numbers or **area**.

Common notion 1. Things which equal the same thing also equal one another.
Common notion 2. If equals are added to equals, then the wholes are equal.
Common notion 3. If equals are subtracted from equals, then the remainders are equal.
Common notion 4. Things which coincide with one another equal one another.
Common notion 5. The whole is greater than the part.

return

Distance

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Definition

The *distance between a point A and a line* ℓ is length of the segment AB where AB is perpendicular to ℓ at B.

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Theorem

Two lines m and ℓ are parallel iff they are *always* the same distance apart.

What does it mean? Is there a better formulation?
Prove it.

Terminology

Modern (new math) text books make a big deal about the difference between congruence and equality. Numbers are central - so equalities are only between numbers while line segments or figures are congruent.

Geometry before Number
Euclid did not have rational numbers as distinct objects - He'd say line segments are congruent (or equal) where we'd say have the same length.

So equality can be replaced by congruence in understanding the common notions.

Ways to think of geometry

Two descriptions of the same dichotomy.

1. synthetic /analytic
2. Euclidian / coordinate (Cartesian)

Ways to think of Mathematics

1. Until the 19th century: Geometry and Arithmetic are based on different intuitions and have separate foundations.
2. Late 19th century: Arithmetic is the most basic; geometry and analysis can be built on arithmetic (proved for geometry by Hilbert).

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We now study the other direction (also do to Hilbert) Analysis (algebra and continuity of the real numbers) can be built from geometry.

Systems of Synthetic Geometry

1. Euclid
2. Hilbert
3. Tarski
4. Birkhoff

Functions and Equivalence Relations

See activity.

Overview on our axiom system

We combine Euclid and Hilbert.
Geometry is fundamental; we define numbers as lengths.

Length

Note that congruence forms an equivalence relation on line segments. We fix a ray ℓ with one end point 0 on ℓ. For each equivalence class of segments, we consider the unique segment $0A$ on ℓ in that class as the representative of that class. We will often denote the class (i.e. the segment $0A$ by a. We say a segment (on any line) CD has length a if $CD \cong 0a$.

In a second stage we will use a to represent both the right hand end and the length - the modern number line.

From geometry to numbers

We want to define the addition and multiplication of numbers. We make three separate steps.

1. identify the collection of all congruent line segments as having a common 'length'. Choose a representative segment OA for this class .
2. define the operation on such representatives.
3. Identify the length of the segment with the end point A. Now the multiplication is on points. And we define the addition and multiplication a little differently.

Defining addition I

Adding line segments

The sum of the line segments AB and AC is the segment AD obtained by extending AB to a straight line and then choose D on AB extended (on the other side of B from A) so that $BD \cong AC$.

Defining addition II

Adding points

Fix a line ℓ and a points 0 on ℓ. We define an operations $+$ on ℓ. Recall that we identify a with the (directed length of) the segment $0a$.

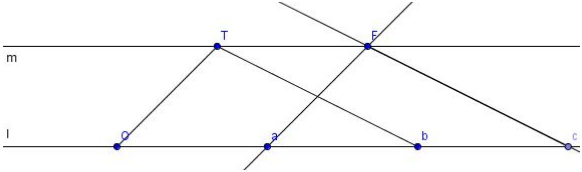
For any points a, b on ℓ,

$$a + b = c$$

if c is constructed as follows.

1. Choose T not on ℓ and m parallel to ℓ through T.
2. Draw $0T$ and BT.
3. Draw a line parallel to $0T$ through a and let it intersect m in F.
4. Draw a line parallel to bT through a and let it intersect ℓ in C.

Diagram for point addition



$$0b \cong ac$$

Activity

Exploring addition of segments and points.

Properties of Addition on points

Why is addition (as defined above) commutative and associative?

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What is the additive inverse of a point of ℓ?

Properties of Addition on points

Why is addition (as defined above) commutative and associative?
What is the additive inverse of a point of ℓ?
Prove addition is associative and commutative with identity element 0. and the additive inverse of a is a' provided that $a'0 \cong 0a$ where a' is on ℓ but on the opposite side of 0 from a.
Here we are implicitly using order axioms.

lunch/Discussion:

lunch/Discussion: How are numbers and geometry being treated than in high school texts

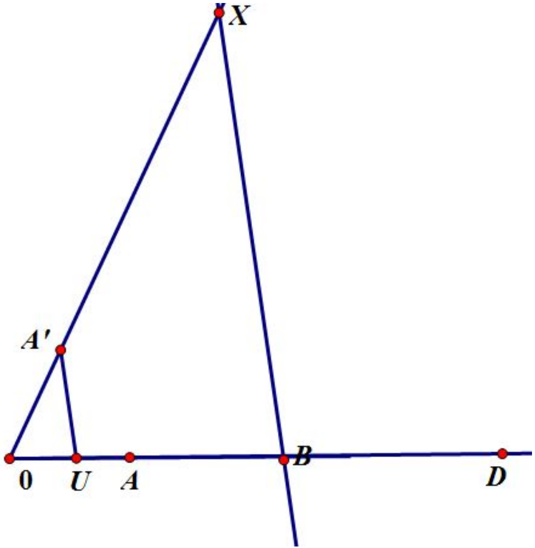
Defining Multiplication 1

Fix points O and U on a line ℓ. (U is for unit).

Multiplying line segments

The product of the line segments OA and OB (on ℓ) is the segment OD obtained as follows.
Draw a line m intersecting OB at O. Lay off a point A' on m so that $OA' \cong OB$. Draw the line $A'U$ and then construct a parallel to $A'U$ through B. Call the intersection of this line with m, X. Now D is the point on ℓ with $OD \cong OX$.

Defining Segment Multiplication diagram

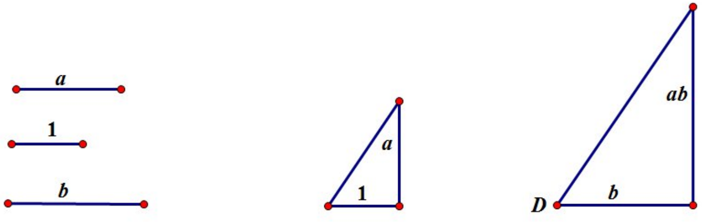


Defining Multiplication II

We will use a less familiar but technically easier version. Consider two segment classes a and b. To define their product, define a right triangle[1] with legs of length a and b. Denote the angle between the hypoteneuse and the side of length a by α. Now construct another right triangle with base of length b with the angle between the hypoteneuse and the side of length b congruent to α. The length of the vertical leg of the triangle is ab.

[1] The right triangle is just for simplicity; we really just need to make the two triangles similar.

Defining point Multiplication diagram



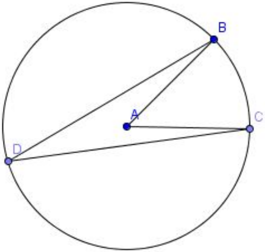
Circles

Do activity on determining circles.

Central and Inscribed Angles

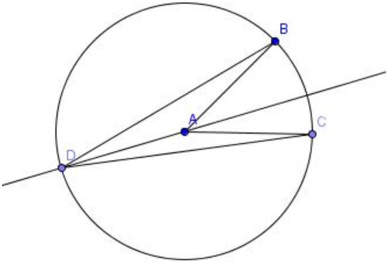
Theorem

[Euclid III.20] **CCSS G-C.2** If a central angle and an inscribed angle cut off the same arc, the inscribed angle is congruent to half the central angle.



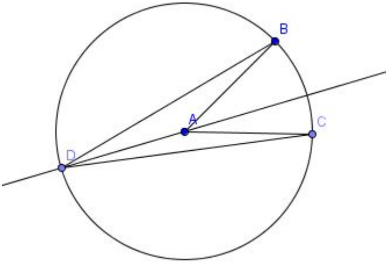
Prove this theorem.

Diagram for proof



Are there any hidden shortcuts hidden in the diagram?

Diagram for proof



Are there any hidden shortcuts hidden in the diagram?

Look at Euclid's proof.

Central and Inscribed Angles

We need proposition 5.8 of Hartshorne, which is a routine high school problem.

CCSS G-C.3 Let $ABCD$ be a quadrilateral. The vertices of $ABCD$ lie on a circle (the ordering of the name of the quadrilateral implies A and B are on the same side of CD) if and only if $\angle DAC \cong \angle DBC$.

More background on Diagrams

Some extracts from the lecture by Jeremy Heis
'Why did geometers stop using diagrams?'
https://webfiles.uci.edu/jheis/www/Heis%2C%20Why%20geometers%20stopped%2C%20ND%20Oct12%20SHORT.pdf

What are the axioms for fields?

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Addition and multiplication are associative and commutative.
There are additive and multiplicative units and inverses.
Multiplication distributes over addition.e

Multiplication

The multiplication defined on points satisfies.

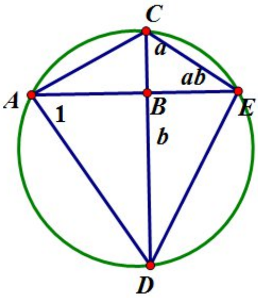
1. For any a, $a \cdot 1 = 1$
2. For any a, b
$$ab = ba.$$
3. For any a, b, c
$$(ab)c = a(bc).$$
4. For any a there is a b with $ab = 1$.

Proving these properties

Think about 1) and 4). Then I will give some hints on 2 and 3.

Commutativity of Multiplication

Given a, b, first make a right triangle $\triangle ABC$ with legs 1 for AB and a for BC. Let α denote $\angle BAC$. Extend BC to D so that BD has length b. Construct DE so that $\angle BDE \cong \angle BAC$ and E lies on AB extended on the other side of B from A. The segment BE has length ab by the definition of multiplication.



Commutativity of Multiplication: finishing the proof

Since $\angle CAB \cong \angle EDB$ by Corollary 40, $ACED$ lie on a circle. Now apply the other direction of Corollary 40 to conclude $\angle DAE \cong \angle DCA$ (as they both cut off arc AD. Now consider the multiplication beginning with triangle $\triangle DAE$ with one leg of length 1 and the other of length b. Then since $\angle DAE \cong \angle DCA$ and one leg opposite $\angle DCA$ has length a, the length of BE is ba. Thus, $ab = ba$.