Geometry, the Common Core, and Proof

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Outline

Agenda

1. G-SRT4 – Context. Proving theorems about similarity
2. Proving that there is a field
3. Areas of parallelograms and triangles
4. lunch/Discussion: Is it rational to fixate on the irrational?
5. Pythagoras, similarity and area
6. reprise irrational numbers and Golden ratio
7. resolving the worries about irrationals

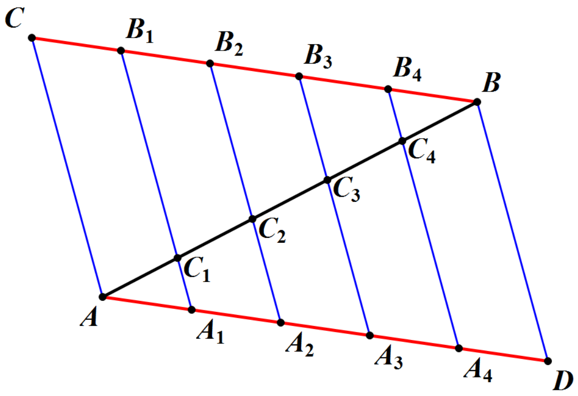
Logistics

G-SRT: Prove theorems involving similarity

4. Prove theorems about triangles. Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.

This is a particularly hard standard to understand. What are we supposed to assume when we give these proofs.
Should the result be proved when the ratio of the side length is irrational?

Activity: Dividing a line into n-parts: Diagram



Side-splitter Theorem

Theorem

Euclid VI.2 **CCSS G-SRT.4** If a line is drawn parallel to the base of triangle the corresponding sides of the two resulting triangles are proportional and conversely.

CME geometry calls this the 'side-splitter theorem' on pages 313 and 315 of CME geometry.

Two steps:

1. Understand the area of triangle geometrically.
2. Transfer this to the formula $A = \frac{bh}{2}$.

The proof uses elementary algebra and numbers; it is formulated in a very different way from Euclid. To prove the side-splitter theorem, we have to introduce numbers.

Towards proving Side-splitter Theorem

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But we won't do it via limits.

Summary -transition

We were able to prove division of lines into n-equal segments using congruence and parallelism.
We *could* apply the side-splitter theorem.

But we haven't proved the full-strength of side-splitter and we don't need it for division into equal segments.

Section 1: From Geometry to Numbers

From geometry to numbers

We want to define the addition and multiplication of numbers. We make three separate steps.

1. identify the collection of all congruent line segments as having a common 'length'. Choose a representative segment OA for this class .
2. define the operation on such representatives.
3. Identify the length of the segment with the end point A. Now the multiplication is on points. And we define the addition and multiplication a little differently.

We focus on step 2; it remains to show the multiplication and addition satisfy the axioms.

Properties of segment addition

Adding line segments

The sum of the line segments AB and AC is the segment AD obtained by extending AB to a straight line and then choose D on AB extended (on the other side of B from A) so that $BD \cong AC$.

Is segment addition associative?
Does it have an additive identity?
Does it have additive inverses?

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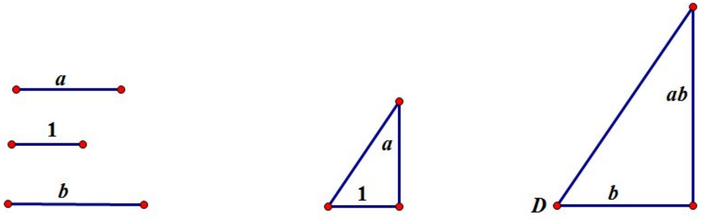
Does it have additive inverses?

Defining Multiplication

Consider two segment classes a and b. To define their product, define a right triangle[1] with legs of length a and b. Denote the angle between the hypoteneuse and the side of length a by α. Now construct another right triangle with base of length b with the angle between the hypoteneuse and the side of length b congruent to α. The length of the vertical leg of the triangle is ab.

[1] The right triangle is just for simplicity; we really just need to make the two triangles similar.

Defining segment Multiplication diagram



Is multiplication just repeated addition?

Activity

We now have two ways in which we can think of the product $3a$.
What are they?

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What are they?

On the one hand, we can think of laying 3 segments of length a end to end.

On the other, we can perform the segment multiplication of a segment of length 3 (i.e. 3 segments of length 1 laid end to end) by the segment of length a.

Prove these are the same.

Discuss: Is multiplication just repeated addition?

Section 2: Proving the field Axioms

What are the axioms for fields?

What are the axioms for fields?

Addition and multiplication are associative and commutative.
There are additive and multiplicative units and inverses.
Multiplication distributes over addition.

Section 3 Reprise: Some theorems on Circles

Circles in the common core

Common Core Standards G-C2, G-C3

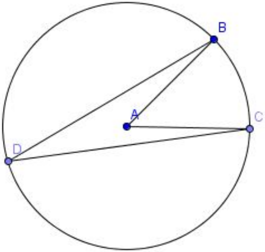
2. Identify and describe relationships among inscribed angles, radii, and chords. Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.

3. Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.

Central and Inscribed Angles

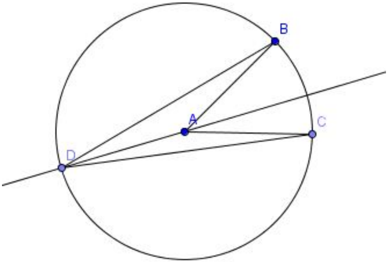
Theorem

[Euclid III.20] **CCSS G-C.2** If a central angle and an inscribed angle cut off the same arc, the inscribed angle is congruent to half the central angle.



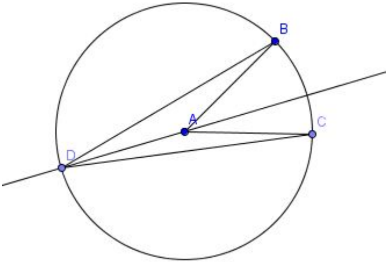
Prove this theorem.

Diagram for proof



Are there any shortcuts hidden in the diagram?

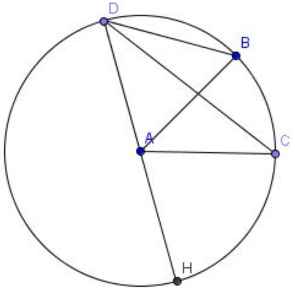
Diagram for proof



Are there any shortcuts hidden in the diagram?

Look at Euclid's proof.

Another Diagram for proof

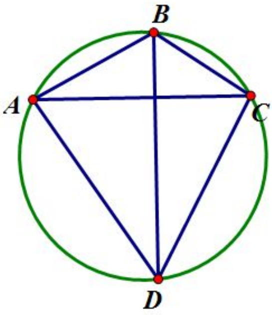


Cyclic Quadrilateral Theorem

CCSS G-C.3

Theorem

Let $ABCD$ be a quadrilateral. The vertices of $ABCD$ lie on a circle (the ordering of the name of the quadrilateral implies A and B are on the same side of CD) if and only if $\angle DAC \cong \angle DBC$.



Multiplication

The multiplication defined on points satisfies.

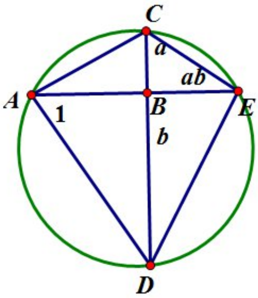
1. For any a, $a \cdot 1 = 1$
2. For any a, b
$$ab = ba.$$
3. For any a, b, c
$$(ab)c = a(bc).$$
4. For any a there is a b with $ab = 1$.

Proving these properties

Do activity multpropact.pdf.

Commutativity of Multiplication

Given a, b, first make a right triangle $\triangle ABC$ with legs 1 for AB and a for BC. Let α denote $\angle BAC$. Extend BC to D so that BD has length b. Construct DE so that $\angle BDE \cong \angle BAC$ and E lies on AB extended on the other side of B from A. The segment BE has length ab by the definition of multiplication.



Commutativity of Multiplication: finishing the proof

Since $\angle CAB \cong \angle EDB$ by the cyclic quadrilateral theorem, $ACED$ lie on a circle. Now apply the other direction of the yclic quadrilateral theorem to conclude $\angle DAE \cong \angle DCA$ (as they both cut off arc AD. Now consider the multiplication beginning with triangle $\triangle DAE$ with one leg of length 1 and the other of length b. Then since $\angle DAE \cong \angle DCA$ and one leg opposite $\angle DCA$ has length a, the length of BE is ba. Thus, $ab = ba$.

Section 4: An Area function

Equal Content

Definition

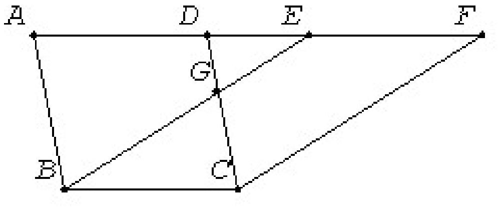
[Equal content] Two figures P, Q have *equal content* if there are figures $P'_1 \ldots P'_n$, $Q'_1 \ldots Q'_n$ such that none of the figures overlap, each P'_i and Q'_i are scissors congruent and $P \cup P'_1 \ldots \cup P'_n$ is scissors congruent with $Q \cup Q'_1 \ldots \cup Q'_n$.

Equal Content for parallelogram

We showed.

[

Euclid I.35, I.38]
Parallelograms on the same base and in the same parallels have the same area. (actually content)
Triangles on the same base and in the same parallels have the same area.



I-35.jpg

Properties of Area

Area Axioms

The following properties of area are used in Euclid I.35 and I.38. We take them from pages 198-199 of CME geometry.

1. Congruent figures have the same area.
2. The area of two 'disjoint' polygons (i.e. meet only in a point or along an edge) is the sum of the two areas of the polygons.
3. Two figures that have equal content have the same area.
4. If one figure is properly contained in another then the area of the difference (which is also a figure) is positive.

Equal area

What are these axioms about? Is there a measure of area?

Equal area

What are these axioms about? Is there a measure of area? Area function to come.

Equal Content as equal area

Clearly equal content satisfies the first 3 conditions for equal area.
It doesn't satisfy the last unless we know there are no figures of zero area.

A crucial lemma Activity

Do the activity– a crucial lemma - Massaging the picture.

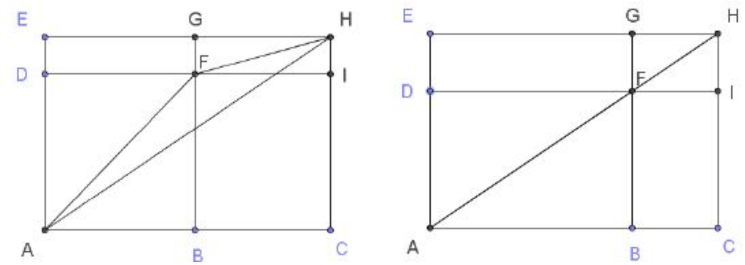
A crucial lemma

Lemma

If two rectangles $ABGE$ and $WXYZ$ have equal content there is a rectangle $ACID$, with the same content as $WXYZ$ and satisfying the following diagram. Further the diagonals of AF and FH are collinear.

Proof. Suppose AB is less than WX and YZ is less than AE. Then make a congruent copy of $WXYZ$ as $ACID$ below. Let F be the intersection of BG and DI. Construct H as the intersection of EG extended and IC extended. Now we prove F lies on AH.

Crucial Lemma: diagram



Crucial Lemma: proof continued

Suppose F does not lie on AH. Subtract $ABFD$ from both rectangles, then $DFGE$ and $BCIF$ have the same area. AF and FH bisect $ABFD$ and $FIHG$ respectively.
So $AFD \cup DFGE \cup FHG$ has the same content as $ABF \cup BCIF \cup FIH$, both being half of rectangle $ACHE$ (Note that the union of the six figures is all of $ACHE$.

Here, $AEHF$ is properly contained in AHE and $ACHF$ properly contains ACH. This contradicts the 4th area axiom; hence F lies on AH.

Crucial Lemma 2

Claim

If $ABGE$ and $ACID$ are as in the diagram (in particular, have the same area), then in segment multiplication $(AB)(BG) = (AC)(CI)$.

Prove this.

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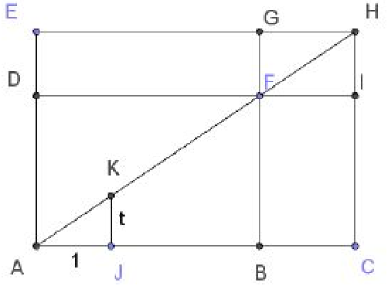
Prove this.

Proof. Let the lengths of AB, BF, AC, CH, JK be represented by a, b, c, d, t respectively and let AJ be 1. Now $ta = b$ and $tc = d$, which leads to $b/a = d/c$ or $ac = bd$, i.e.$(AB)(BG) = (AC)(CI)$.

Diagram for Claim



By congruence, we have $(AE)(AB) = (WZ)(XY)$ as required.

The area function

Definition

The area of a square 1 unit on a side is (segment arithmetic) product of its base times its height, that is one square unit.

Theorem

The area of a rectangle is the (segment arithmetic) product of its base times its height.

Proof. Note that for rectangles that have integer lengths this follow from Activity on multiplication as repeated addition. For an arbitrary rectangle with side length c and d, apply the identity law for multiplication and associativity.

What is there still to prove?

Do activity fndef.pdf

Towards the area formula for triangles

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Exercise

Draw a scalene triangle such that only one of the three altitudes lies within the triangle. Compute the area for each choice of the base as b (and the corresponding altitude as h).

Towards the area formula for triangles

Theorem

Any of the three choices of base for a triangle give the same value for the product of the base and the height.

Towards the area formula for triangles

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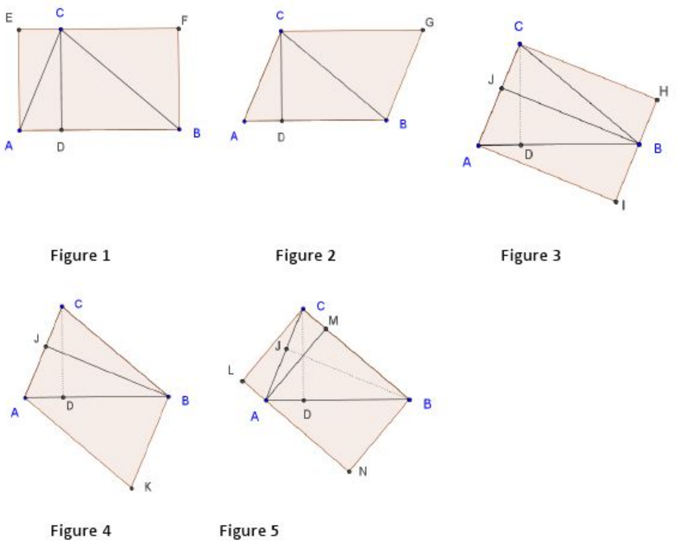
Proof. Consider the triangle ABC is figure 1. The rectangles in figures 1,3, and 5 are easily seen to be scissors congruent. By the Claim, each product of height and base for the three triangles is the same. That is, $(AB)(CD) = (AC)(BJ) = (BC)(AM)$. But these are the three choices of base/altitude pair for the triangle ABC.

Diagram for the area formula for triangles



Section 5: The side-splitter theorem

Basic Tool

We have shown.

Assumption

The area of a triangle with base b and height h is given by $\frac{1}{2}bh$ using segment multiplication.

Proving the Sidesplitter

side-splitter activity

Resolving Raimi's worry

Using the formula $A = \frac{1}{2}bh$ we have shown.

Theorem

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The theorem holds for all lengths of sides of a triangle that are in the plane.

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The theorem holds for all lengths of sides of a triangle that are in the plane.

But there may be lengths (e.g. π) which aren't there. (E.g. if the underlying field is the real algebraic numbers).
So CCSM was right to have both segment length and arc length as primitive terms.

Resolving Raimi's worry

Theorem

If two triangles have the same height, the ratio of their areas equals the ratio of the length of their corresponding bases.

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Section 6 The Pythagorean Theorem

Pythagorean Theorem

Consider various proofs:

1. Look at the Garfield paper. (Draw the diagram that got blacked out by 137 years.)
2. Prove using similar triangles. (Hint: draw a perpendicular from the hypoteneuse to the opposite vertex; follow your nose. `http://aleph0.clarku.edu/~djoyce/java/elements/bookVI/propVI31.html`
3. The standard Euclid proof `http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI47.html`
4. your favorite
5. 98 proofs `http://cut-the-knot.org/pythagoras/#pappa`

Which (proofs) are in your textbook?
Note that each proof uses either *area* or *similarity*.

Section 7 Irrational Numbers

CCSSM on irrationals I

The Number System 8.NS Know that there are numbers that are not rational, and approximate them by rational numbers.

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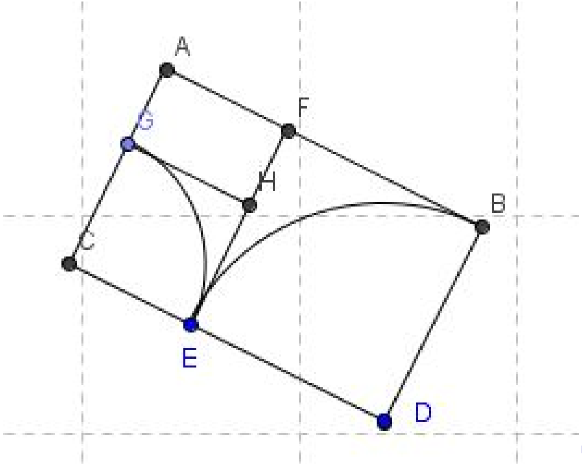
CCSSM on irrationals II

Expressions and Equations 8.EE Work with radicals and integer exponents. 1. Know and apply the properties of integer exponents

Use square root and cube root symbols to represent solutions to equations of the form x2 = p and x3 = p, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.

The Golden Ratio

Consider the diagram below where $ABDC$ is a rectangle, $BD = DE$, $CE = CG$ and GH is parallel to CD and $\frac{CD}{BD} = \frac{AC}{EC} = \frac{AF}{AG} = \phi$.



Irrationality of the Golden Ratio

Suppose $\phi = \frac{CD}{BD} = \frac{m}{n}$ for some integers m, n. How can you now express each of the lengths CD, BD, CE, AC, GH, AG?

Is the choice of such m, n possible?

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Howard on Fibonacci
http://homepages.math.uic.edu/~howard/spirals/