PERSPECTIVES ON EXPANSIONS

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SETTING

M is a structure for a language L, A is a subset of M.

$L^* = L(P)$ is the expansion of L by one unary predicate and (M, A) is the L^*-structure where P is interpreted by A.

When does (M, A) have the same stability class as M?

PERSPECTIVES

I. Analysis of Arbitrary expansions

II. Constructing Expansions:

What hath Hrushovski wrought?

TWO FACTORS

What structure does M ‘induce’ on A?

How does A ‘sit in’ M?

INDUCE

The basic formulas induced on A can be:

L^*: the traces on A of parameter free L-formulas (*induced structure*);

$L^\#$: the traces on A of parameter free $L(P)$-formulas ($L^\#$-*induced structure*, $A^\#$);

EXAMPLES

Form a structure M with a two sorted universe:

1. The complex numbers.

2. A fibering over the complex numbers.

Let N extend M by putting one new point in the fiber over a if and only if a is a real number.

Now M and N are isomorphic and are ω-stable nfcp. But the structure (N, M) is unstable.

The $*$-induced structure on M is stable since in fact no new sets are definable.

In the $\#$-induced structure

$$(\exists x)E(x, y) \wedge x \notin P$$

defines the reals so the $\#$-induced structure is unstable.

SITS

Definition 1 M *is* ω*-saturated over* A, *(*A *is small in* M*), if for every* $\overline{a} \in M - A$, *every* L*-type* $p \in S(\overline{a}A)$ *is realized in* M.

Definition 2 *1. The set A is* weakly benign *in M if for every $\alpha, \beta \in M$ if:*

$$\mathsf{stp}(\alpha/A) = \mathsf{stp}(\beta/A)$$

implies

$$\mathsf{tp}_*(\alpha/A) = \mathsf{tp}_*(\beta/A).$$

2. (M, A) is *uniformly weakly benign* if every (N, B) which is $L(P)$-elementarily equivalent to (M, A) is weakly benign.

SUFFICIENT CONDITIONS

Baldwin-Benedikt

Theorem 3 *If M is stable and I is a set of indiscernibles so that (M, I) is small, then (M, A) is stable.*

Explaining Baldwin-Benedikt, and linking with Poizat:

Casanovas-Ziegler prove:

Theorem 4 *If* (M, A) *has the nfcp (over* A*), is small, and the* $*$*-induced theory on* A *is stable then* (M, A) *is stable.*

Extending Casanovas-Ziegler,

Baizhanov-Baldwin prove:

Theorem 5 *If* (M, A) *is uniformly weakly benign and the #-induced theory on* A *is stable then* (M, A) *is stable.*

Theorem 6 implies Theorem 5 by Polkowska's remark:

Remark 6 *If* (M, A) *is small and* M *has nfcp then the* $*$*-induced and* $\#$*-induced theories on* A *are the same.*

Bouscaren showed (in our language):

Theorem 7 *If N is superstable and $M \prec N$, then (N, M) is weakly benign.*

Baizhanov, Baldwin, Shelah showed:

Theorem 8 *If M is superstable (M, A) is uniformly weakly benign for any A.*

Question 9 *If M is stable must (M, A) be uniformly weakly benign for any A?*

Question 10 *Is there a stable structure M and an infinite set of indiscernibles I such I is not indiscernible in (M, I)?*

Polowska gave conditions ‘bounded PAC’ on (M, A) so that

Th$_*$(A) is simple and (M, A) is simple.

Question 11 *If* (M, A) *is small and* M *has nfcp and* $\mathrm{Th}_*(\mathrm{A})$ *is simple must* (M, A) *be simple?*

THE HRUSHOVSKI MACHINE

INPUT: $\boldsymbol{K}_0, \delta, \mu$

DESIRED OUTPUT: Nice theory

ACTUAL OUTPUT:

Nice structure: the generic

An infinitarily defined class

What is a ‘first order structure’?

Possible meaning:

The class of existentially closed models (in appropriate language) is first order.

PROBLEM

In the *ab initio* or free amalgamation cases the generic is ω-saturated and its theory is the answer. Even so,

the class of existentially closed models may not be elementary (in given language).

In more complicated situations, the generic may not be saturated.

FRAMEWORKS

Robinson Theory

CATS

AEC/Quasiminimal excellence

MAKING EC MODELS ELEMENTARY

I. Axiomatize more carefully.

II. Adjust the language.

QUANTIFIER COMPLEXITY

Definition 12 T *is* model complete *if every formula* $\phi(\overline{x})$ *is equivalent to an existential formula.*

Definition 13 T *is* nearly model complete *if every formula* $\phi(\overline{x})$ *is equivalent to a* Boolean combination *of existential formulas.*

LINDSTROMS'S LITTLE THEOREM

Theorem 14 *If T is π_2-axiomatizable and categorical in some infinite cardinality then T is model complete.*

Hrushovski-like Constructions

theory	axioms	complexity
sm set/ fusions	π_2	m.c.
rank ω bicol field	π_2	n.m.c.
rank 2 bicol field	π_2	m.c.
Spencer-Shelah	π_2	n.m.c.

The original axiomatizations of the sm set and of fusions (Hrushovski) and of the $n^{-\alpha}$-random graph (Baldwin-Shelah) were π_3. The proofs of the rank 2 bicolored field (Baldwin-Holland) did not find axioms explicit axioms. But Holland did. Axioms were explicitly provided for bicolored fields by Baudisch, Martin-Pizzaro, and Ziegler.

Holland, Laskowski and Baldwin-Holland (respectively) made the improvements.

Definition 15 *A formula* $\phi(x_1, \ldots, x_k)$ *has* exactly rank $k-1$ *if for every generic (over the parameters of* ϕ*) solution* $\overline{a} = \langle a_1, \ldots, a_k \rangle$ *of* $\phi(x_1, \ldots, x_k)$*,* $R_M(\overline{a}) = k - 1$ *and any proper subsequence of* $\overline{a}$ *is independent.*

Assumption 16 *The underlying theory T_f satisfies the following condition: If $\psi_i(\overline{x}, y)$ for $i < m$ are a finite set of $k+1$-ary formulas such that for some g, for each $i < m$, $\psi_i(\overline{x}, g)$ has rank at most $k-1$, there is a formula $\phi(\overline{x}, y)$ such that $\phi(\overline{x}, g)$ has exactly rank $k-1$ and for each $i < m$,*

$$R_M(\phi(\overline{x}, g) \wedge \psi_i(\overline{x}, g)) < k - 1.$$

Theorem 17 *If T_f is strongly minimal with elimination of imaginaries and dmp and satisfies Assumption 16 then T_k^μ is model complete. In particular, the rank k bicolored fields are model complete.*

Example

If when trying to collapse the bicolored field (or a fusion), the function μ is chosen badly, the

generic is not saturated

and its theory is not even nearly model complete.

If the generic is not saturated, can we explain a sense in which it is nice.

Perhaps get a fallback-theorem?

ABSTRACT ELEMENTARY CLASSES

A class of L-structures, $(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$, is said to be an *abstract elementary class: AEC* if both $\boldsymbol{K}$ and the binary relation $\preceq_{\boldsymbol{K}}$ are closed under isomorphism and satisfy the Jonssón conditions plus

If $A, B, C \in \boldsymbol{K}$, $A \preceq_{\boldsymbol{K}} C$, $B \preceq_{\boldsymbol{K}} C$ and $A \subseteq B$ then $A \preceq_{\boldsymbol{K}} B$;

More careful formulation of unions of chains;

Existence of Lowenheim number.

QUASIMINIMALITY I

Trial Definition M is ‘*quasiminimal*’ if every first order ($L_{\omega_1,\omega}$?) definable subset of M is countable or cocountable.

$a \in \mathrm{acl}'(\mathrm{X})$ if there is a first order formula with countably many solutions over X which is satisfied by a.

Exercise ? If f takes X to Y is an elementary isomorphism, f extends to an elementary isomorphism from $\mathrm{acl}'(\mathrm{X})$ to $\mathrm{acl}'(\mathrm{Y})$.

QUASIMINIMAL EXCELLENCE

A class $(\boldsymbol{K}, \mathrm{cl})$ is *quasiminimal excellent* if it admits a combinatorial geometry which satisfies on each $M \in \boldsymbol{K}$

there is a unique type of a basis,

a technical homogeneity condition:
$\aleph_0$-homogeneity over $\emptyset$ and over models.

Definition 18 *Let* $C \subseteq H \in \boldsymbol{K}$ *and let* X *be a finite subset of* H. *We say* $\mathrm{tp}_{\mathrm{qf}}(X/C)$ *is* defined over *the finite* C_0 *contained in* C *if: for every* G*-partial monomorphism* f *mapping* X *into* H'*, for every* G*-partial monomorphism* f_1 *mapping* C *into* H'*, if* $f \cup (f_1|C_0)$ *is a* G*-partial monomorphism,* $f \cup f_1$ *is also a* G*-partial monomorphism.*

In the following definition it is essential that $\subset$ be understood as *proper* subset.

Definition 19 *1. For any* Y, $\mathrm{cl}^{-}(Y) = \cup_{X \subset Y} \mathrm{cl}(X)$.

2. We call C *(the union of)* an n-dimensional cl-independent system *if* $C = \mathrm{cl}^{-}(Z)$ *and* Z *is an independent set of cardinality* n.

[Condition IV: Quasiminimal Excellence] Let $G \subseteq H, H' \in \boldsymbol{K}$ with G empty or in $\boldsymbol{K}$. Suppose $Z \subset H - G$ is an n-dimensional independent system, $C = \mathrm{cl}^{-}(Z)$, and X is a finite subset of $\mathrm{cl}(Z)$. Then there is a finite C_0 contained in C such that $\mathrm{tp}_{\mathrm{qf}}(X/C)$ is defined over C_0.

Excellence yields:

Lemma 20 *An isomorphism between independent X and Y extends to an isomorphism of* $\mathrm{cl}(X)$ *and* $\mathrm{cl}(Y)$.

This gives categoricity in all uncountable powers if the closure of finite sets is countable.

AEC

$\boldsymbol{K}$ from Hrushovski construction.

$d_M(A) = \text{inf}\{\delta(B) : A \subseteq B \subset_\omega M\}$

$M \preceq_{\boldsymbol{K}} N$ iff for every finite $A \subset M$

$$d_M(A) = d_N(A)$$

FACT: $(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$ is an aec.

Quasi Min Excell of Hrushovski Construction

$a \in \mathrm{cl}_M(X)$ iff $d_M(a/X) = 0$.

Note closure is $L_{\omega_1,\omega}$-definable and

$a \in \mathrm{cl}_M(X)$ implies $\mathrm{tp}(a/X_0)$ has finitely many solutions for some finite $X_0 \subseteq X$.

Fact: $(\boldsymbol{K}, \mathrm{cl}_M)$ is a quasiminimal excellent class.

Application

For bicolored fields and any μ the Scott sentence of the generic model is categorical in all cardinalities.

Compare with work of Villaveces and Zambrano.

Zilber Variation

In defining the class $\boldsymbol{K}$ for the Hrushovski construction allow sentences of $L_{\omega_1,\omega}$.

Specifically, fix $(Z, +)$.

It still works!

Pseudo-exponentiation et al

Connecting the two parts of the talk.

When the second approach does not yield a 'first order structure',

study the theory of the generic by the first approach.

Bibliography

See accompanying file.