Why AEC?

Categoricity and Complex Exponentiation

Excellence–Generalized Amalgamation

Eventual Categoricity

Core Mathematics again

Abstract Elementary Classes Motivations and Directions

John T. Baldwin

University of Illinois at Chicago

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Topics

1 Why AEC?

2 Categoricity and Complex Exponentiation

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4 Eventual Categoricity

5 Core Mathematics again

Two Goals

General

Can we extend the methods of first order stability theory to generalized logics – e.g. $L_{\omega_1,\omega}$?

Special

Can the model theory of infinitary logic solve 'mathematical problems' (as the model theory of first order logic has)?

A background principle

slogan

To study a structure A, study $\mathrm{Th}(A)$.

e.g.

The theory of algebraically closed fields to investigate $(\mathcal{C}, +, \cdot)$.

The theory of real closed fields to investigate $(\mathcal{R}, +, \cdot)$.

ABSTRACT ELEMENTARY CLASSES defined

Definition

A class of L-structures, $(\mathbf{K}, \prec_{\mathbf{K}})$, is said to be an <u>abstract elementary class: AEC</u> if both $\mathbf{K}$ and the binary relation $\prec_{\mathbf{K}}$ are closed under isomorphism and satisfy the following conditions.

- **A1**. If $M \prec_{\mathbf{K}} N$ then $M \subseteq N$.
- **A2**. $\prec_{\mathbf{K}}$ is a partial order on $\mathbf{K}$.
- **A3**. If $\langle A_i : i < \delta \rangle$ is $\prec_{\mathbf{K}}$-increasing chain:
  1. $\bigcup_{i<\delta} A_i \in \mathbf{K}$;
  2. for each $j < \delta$, $A_j \prec_{\mathbf{K}} \bigcup_{i<\delta} A_i$
  3. if each $A_i \prec_{\mathbf{K}} M \in \mathbf{K}$ then $\bigcup_{i<\delta} A_i \prec_{\mathbf{K}} M$.

- **A4**. If $A, B, C \in \mathbf{K}$, $A \prec_{\mathbf{K}} C$, $B \prec_{\mathbf{K}} C$ and $A \subseteq B$ then $A \prec_{\mathbf{K}} B$.
- **A5**. There is a Löwenheim-Skolem number $\mathrm{LS}(\mathbf{K})$ such that if $A \subseteq B \in K$ there is a $A' \in \mathbf{K}$ with $A \subseteq A' \prec_{\mathbf{K}} B$ and
  $|A'| < \mathrm{LS}(\mathbf{K}) + |\mathrm{A}|$.

Examples

1. First order complete theories with $\prec_{\mathbf{K}}$ as elementary submodel.
2. Models of $\forall\exists$-first order theories with $\prec_{\mathbf{K}}$ as substructure.
3. L^n-sentences with L^n-elementary submodel.
4. Varieties and Universal Horn Classes with $\prec_{\mathbf{K}}$ as substructure.
5. Models of sentences of $L_{\kappa,\omega}$ with $\prec_{\mathbf{K}}$ as: elementary in an appropriate fragment.
6. Models of sentences of $L_{\kappa,\omega}(Q)$ with $\prec_{\mathbf{K}}$ carefully chosen.
7. Robinson Theories with Δ-submodel
8. 'The Hrushovski Construction' with strong submodel

GÖDEL PHENOMENA

It follows from Gödels work in the 30's that:

1. The collections of sentences true in $(Z, +, \cdot, 0, 1)$ is undecidable.
2. There are definable subsets of $(Z, +, \cdot, 0, 1)$ which require arbitrarily many alternations of quantifiers. (Wild)

COMPLEX EXPONENTIATION

Consider the structure $(C, +, \cdot, e^x, 0, 1)$.
It is Godelian.
The integers are defined as $\{a : e^a = 1\}$.
The first order theory is undecidable and ‘wild’.

ZILBER'S INSIGHT

Maybe Z is the source of all the difficulty. Fix Z by adding the axiom:

$$(\forall x)e^x = 1 \rightarrow \bigvee_{n \in Z} x = 2n\pi.$$

Model Theory of $\mathcal{C}$

The first order theory of the complex field is categorical and admits quantifier elimination.

Model theoretic approaches based on Shelah's theory of orthogonality have led to advances such as Hrushovski's proof of the geometric Mordell-Lang conjecture.

The first order theory of complex exponentiation is model theoretically intractable.

Zilber conjectures complex exponentiation has a categorical axiomatization in infinitary logic.

MORLEY'S THEOREM

Theorem

If a countable first order theory is categorical in one uncountable cardinal it is categorical in all uncountable cardinals.

GEOMETRIES

Definition. A pregeometry is a set G together with a dependence relation

$$cl : \mathcal{P}(G) \rightarrow \mathcal{P}(G)$$

satisfying the following axioms.

A1. $cl(X) = \bigcup\{cl(X') : X' \subseteq_{fin} X\}$
A2. $X \subseteq cl(X)$
A3. $cl(cl(X)) = cl(X)$
A4. If $a \in cl(Xb)$ and $a \notin cl(X)$, then $b \in cl(Xa)$.
If points are closed the structure is called a geometry.

CLASSSIFYING GEOMETRIES

Geometries are classified as: trivial, locally modular, non-locally modular.

Zilber had conjectured that each non-locally modular geometry of a strongly minimal set was 'essentially' the geometry of an algebraically closed field.

Zilber now proposes to use Hrushovski's construction which gave counterexamples to this conjecture and to provide an infinitary categorical theory of complex exponentiation.

STRONGLY MINIMAL I

Definition

M is strongly minimal if every first order definable subset of any elementary extension M' of M is finite or cofinite.

Every strongly minimal set is categorical in all uncountable powers.
The complex field is strongly minimal.

STRONGLY MINIMAL II

Lemma.
$a \in \mathrm{acl}(B)$ if $\phi(a, \mathbf{b})$ and $\phi(x, \mathbf{b})$ has only finitely many solutions.
A complete theory T is strongly minimal if and only if it has infinite models and

1. algebraic closure induces a pregeometry on models of T;
2. any bijection between *acl*-bases for models of T is an elementary map.

QUASIMINIMALITY I

Trial Definition M is 'quasiminimal' if every first order ($L_{\omega_1,\omega}$?) definable subset of M is countable or cocountable.

$a \in \mathrm{acl}'(X)$ if there is a first order formula with **countably many** solutions over X which is satisfied by a.

Exercise ? If f takes X to Y is an elementary isomorphism, f extends to an elementary isomorphism from $\mathrm{acl}'(X)$ to $\mathrm{acl}'(Y)$.

QUASIMINIMAL EXCELLENCE

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1. there is a unique type of a basis,
2. a technical homogeneity condition: $\aleph_0$-homogeneity over $\emptyset$ and over models.
3. and the ‘excellence condition’ which follows.

Defining Excellence: Easy Case

In the following definition it is essential that $\subset$ be understood as <u>proper</u> subset.

Definition

1. For any Y, $\mathrm{cl}^{-}(Y) = \bigcup_{X \subset Y} \mathrm{cl}(X)$.
2. We call C (the union of) <u>an n-dimensional cl-independent system</u> if $C = \mathrm{cl}^{-}(Z)$ and Z is an independent set of cardinality n.

Let say $\mathrm{tp}_{\mathrm{qf}}(X/C)$ is defined over the finite C_0 contained in C if it is determined by its restriction to C_0.

[Quasiminimal Excellence] Let $G \subseteq H, H' \in \mathbf{K}$ with G empty or in $\mathbf{K}$. Suppose $Z \subset H - G$ is an n-dimensional independent system, $C = \mathrm{cl}^{-}(Z)$, and X is a finite subset of $\mathrm{cl}(Z)$. Then there is a finite C_0 contained in C such that $\mathrm{tp}_{\mathrm{qf}}(X/C)$ is defined over C_0.

3 and 4 amalgamation

We use two slides in another format.

EXCELLENCE IMPLIES CATEGORICITY

Excellence implies by a direct limit argument:

Lemma

An isomorphism between independent X and Y extends to an isomorphism of $\mathrm{cl}(X)$ and $\mathrm{cl}(Y)$.

This gives categoricity in all uncountable powers if the closure of finite sets is countable.

CATEGORICITY

Theorem Suppose the quasiminimal excellent (I-IV) class **K** is axiomatized by a sentence Σ of $L_{\omega_1,\omega}$, and the relations $y \in \mathrm{cl}(x_1, \ldots x_n)$ are $L_{\omega_1,\omega}$-definable.

Then, for any infinite κ there is a unique structure in **K** of cardinality κ which satisfies the countable closure property.

NOTE BENE: The categorical class could be axiomatized in $L_{\omega_1,\omega}(Q)$. But, the categoricity result does not depend on any such axiomatization.

ZILBER'S PROGRAM FOR $(\mathcal{C}, +, \cdot, \exp)$

Goal: Realize $(\mathcal{C}, +, \cdot, \exp)$ as a model of an $L_{\omega_1,\omega}$-sentence discovered by the Hrushovski construction.

Objective A

Expand $(\mathcal{C}, +, \cdot)$ by a unary function f which behaves like exponentiation using a Hrushovski-like dimension function. Prove some $L_{\omega_1,\omega}$-sentence Σ satisfied by $(\mathcal{C}, +, \cdot, f)$ is categorical and has quantifier elimination.

Objective B

Prove $(\mathcal{C}, +, \cdot, \exp)$ is a model of the sentence Σ found in Objective A.

Excellence for $L_{\omega_1,\omega}$

Any κ-categorical sentence of $L_{\omega_1,\omega}$ can be replaced (for categoricity purposes) by considering the atomic models of a first order theory. ($EC(T, Atomic)$-class)

Shelah defined a notion of excellence; Zilber's is the 'rank one' case.

ω-stabilty

$(\mathbf{k}, \prec_{\mathbf{K}})$ is the class of atomic models of a first order theory under elementary submodel.

Definitions

$p \in S_{at}(A)$ if $a \models p$ implies Aa is atomic.

$\mathbf{K}$ is ω-stable if for every countable model M, $S_{at}(M)$ is countable.

Theorem

[Keisler/Shelah]
$(2^{\aleph_0} < 2^{\aleph_1})$ If $\mathbf{K}$ has $< 2^{\aleph_1}$ models of cardinality $\aleph_1$, then $\mathbf{K}$ is ω-stable.

EARLIER RESULTS

Theorem (Shelah 1983)

If **K** *is an excellent* $EC(T, Atomic)$*-class then if it categorical in one uncountable cardinal, it is categorical in all uncountable cardinals.*

Theorem (Shelah 1983)

If $2^{\aleph_n} < 2^{\aleph_{n+1}}$ *and an* $EC(T, Atomic)$*-class* **K** *is categorical in all* $\aleph_n$ *for all* $n < \omega$*, then it is excellent.*

Splitting

Definition

A complete type p over A splits over $B \subset A$ if there are $\mathbf{b}, \mathbf{c} \in A$ which realize the same type over B and a formula $\phi(\mathbf{x}, \mathbf{y})$ with $\phi(\mathbf{x}, \mathbf{b}) \in p$ and $\neg\phi(\mathbf{x}, \mathbf{c}) \in p$.

Splitting

Definition

Let ABC be atomic. We write $A \underset{C}{\smile\!\!\!\!|} B$ and say A is free or independent from B over C if for any finite sequence $\mathbf{a}$ from A, $\mathrm{tp}(\mathbf{a}/B)$ does not split over some finite subset of C.

For ω-stable atomic classes, one gets all the nice properties of forking with one crucial restriction.
Only types over models (or good sets) behave really well.

Goodness

A set A is good if the isolated types are dense in $S_{at}(A)$.

If A is countable and good there is a prime model over A.

Excellence

The class **K** is excellent if for every independent system of countable sets: $\langle M_s : s \subset n \rangle$,

$$\bigcup_{s \subset n} M_s$$

is good.

Consequences of Excellence

If **K** is excellent and has an uncountable model then **K** has models in every uncountable power.
Why? Show two steps.

The role of Set theory

first order logic

One of the principal effects of stability theory is to separate axiomatic set theory from model theory.

infinitary logic

A hope is that for the study of excellence one needs only minimal additons to ZFC - e.g. $2^\lambda < 2^{\lambda^+}$ Some addtions are necessary. It is consistent that an $\aleph_1$-categorical AEC is not ω-stable.

Getting Excellence

Theorem Desired: 2^κ is increasing

If $I(\aleph_n, \mathbf{K}) < 2^{\aleph_n}$ then $\mathbf{K}$ is excellent.

Actual Theorems: 2^κ is increasing

1. If $I(\aleph_n, \mathbf{K}) < \mu(n)$ then $\mathbf{K}$ is excellent, where $\mu(n)$ has a complicated definition.

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1. If $I(\aleph_n, \mathbf{K}) < \mu(n)$ then $\mathbf{K}$ is excellent, where $\mu(n)$ has a complicated definition.
2. If the ideal of small subsets of λ^+ is not λ^{++}-saturated and $I(\aleph_n, \mathbf{K}) < 2^{\aleph_n}$ then $\mathbf{K}$ is excellent.

Getting Excellence

Stumbling block

Under appropriate set theory and a number of technical model theoretic assumptions:

if there are 'few' models in $\aleph_{n+2}$ then every independent two system in $\aleph_n$ is an amalgamation base.

Eventual Categoricity: Context

Conjecture

Let X be a class of cardinals in which a reasonably defined class is categorical.

Exactly one of X and its complement is cofinal.

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(Note: So, PC-classes are not 'reasonable'. The class:

$$\{(M, X) : 2^{|X|} \geq |M|\}$$

is categorical only in strong limit cardinals.

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is categorical only in strong limit cardinals.

Of course, it is only interesting when $\mathbf{K}$ has arbitrarily large models – EM methods are applicable.

Resolution and Generalization

We outline the proof of the 'successor conjecture' for AEC with amalgamation:

If an AEC with ap is categorical on a class of successor cardinals then it is eventually categorical.

Tameness

Definition

1. We say **K** is (χ, μ)-weakly tame if for any saturated $N \in \mathbf{K}$ with $|N| = \mu$ if $p, q, \in \mathcal{S}(N)$ and for every $N_0 \leq N$ with $|N_0| \leq \chi$, $p \restriction N_0 = q \restriction N_0$ then $q = p$.
2. We say **K** is (χ, μ)-tame if the previous condition holds for all N with cardinality μ.

Tameness-Algebraic form

Suppose **K** has the amalgamation property.

K is (χ, μ)-tame if for any model M of cardinality μ and any $a, b \in \mathcal{M}$:

Tameness-Algebraic form

Suppose **K** has the amalgamation property.

K is (χ, μ)-tame if for any model M of cardinality μ and any $a, b \in \mathcal{M}$:

If for every $N \prec_{\mathbf{K}} M$ with $|N| \leq \chi$ there exists $\alpha \in \mathrm{aut}_{\mathrm{N}}(\mathcal{M})$ with $\alpha(a) = b$,

then there exists $\alpha \in \mathrm{aut}_{\mathrm{M}}(\mathcal{M})$ with $\alpha(a) = b$.

Theorem (Grossberg-Vandieren)

If $\mathbf{K}$ *is* λ^+*-categorical and* $(<\lambda, \infty)$*-tame then* $\mathbf{K}$ *is categorical in all* $\theta \geq \lambda^+$.

Downward Categoricity

Suppose the vocabulary is countable; $H_1 = \beth_{(2^\omega)^+}$.

Theorem

If the AEC **K** *has*

1. *ap*
2. *jep*
3. *is categorical in a successor cardinal* λ^+ *and* $\lambda > H_1$

then **K** *is categorical in every* θ *with* $H_1 \leq \theta \leq \lambda$.

Shelah with some minor improvements/corrections by Hyttinen and myself and using the last theorem from Grossberg-VanDieren.

proof outline

Categoricity in λ implies

1. stability below λ (any λ);

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Categoricity in λ implies

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Categoricity in λ implies

1. stability below λ (any λ);
2. categoricity model saturated (λ regular);
3. **K** is $(H_1, < \lambda)$-tame (λ regular);
4. $|N| \geq H_1$ implies N is H_1-saturated (λ-regular and tameness);
5. No two cardinal model in H_1 (λ-successor);
6. Categoricity above H_1 (categoricity theorem for tame classes).

UNIVERSAL COVERS

Study the model theory of the exact sequence:

$$0 \to K \to V \to A \to 0. \tag{1}$$

Contexts

- A is $\aleph_1$-free; K is Z.
- A is a semiabelian variety; K is Z^d.

$\aleph_1$-free

Baldwin and Shelah have used $\aleph_1$-free but not free A to construct various examples of non-tame abstract elementary classes.

Semi-Abelian Varieties

Definition An algebraic group $A(\mathcal{C})$ is a **semi-abelian** variety if there is short exact sequence

$$0 \to Z^N \to \mathcal{C}^d \to A(\mathcal{C}) \to 1. \tag{2}$$

where the map from $\mathcal{C}^d$ to $A(\mathcal{C})$ is an analytic homomorphism and $d \leq N \leq 2d$.

UNIVERSAL COVERS

When is the exact sequence:

$$0 \to Z^N \to V \to A \to 1. \tag{3}$$

categorical where V is a $\mathbb{Q}$ vector space and A is a semi-abelian variety?

Can be viewed as an expansion of V and there is a combinatorial geometry given by:

$$\mathrm{cl}(X) = \ln(\mathrm{acl}(\exp(X)))$$

Geometry from Model Theory

Zilber has shown equivalence between certain 'arithmetic' statements about Abelian varieties and model theoretic properties of the associated AEC –categoricity below $\aleph_\omega$.

The equivalence depends on weak extensions of set theory and Shelah's categoricity transfer theorem.

A Little More Detail

These statements are variants of the 'thumbtack lemma' for the variety A, generalizing $\mathcal{C}^*$.

There are a semiabelian varieties which are known not to satisfy the conditions.

There are a semiabelian varieties for which these conditions are an open question.

Another direction is to try to adapt abstract arguments of Grossberg and Kolesnikov to get tameness from Hrushovski constructions or at least in these specific semiabelian contexts

Infinitary Logic and Core Mathematics

The work on pseudoexponentiation raises significant questions in complex variable theory and algebraic geometry.

The work categoricity of semi-abelian varieties moves to a different level. It actually finds 'arithmetic' consequences of assuming categoricity beyond $\aleph_1$ of the infinitary theory of certain varieties.